Rotational Dynamics

Rigid body

It is a solid body in which the particles are arranged compactly so that the interparticle distance is small and fixed and their positions are not disturbed by any external forces applied on it.

No real body is truly rigid.

Translatory motion - Same displacement, Different velocity
Rotational motion - Same angular velocity (ω), Different angular displacement (θ)

Equations of Angular Motion

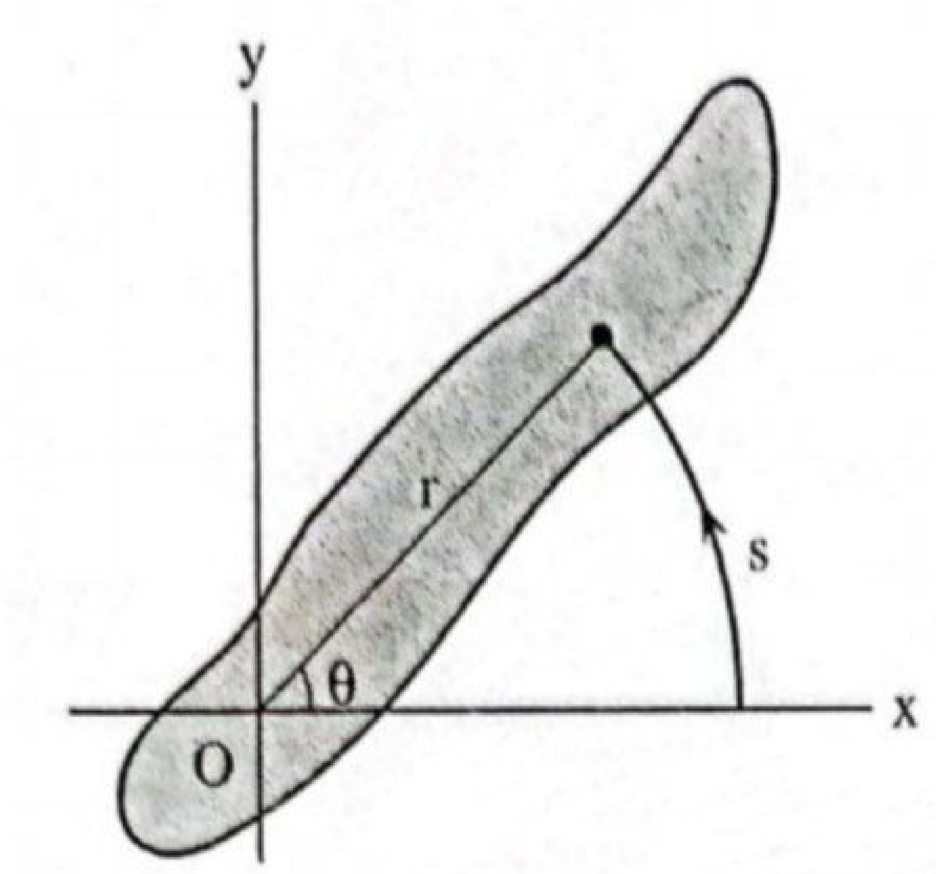


Fig. 1 A rigid body rotating about an axis through O

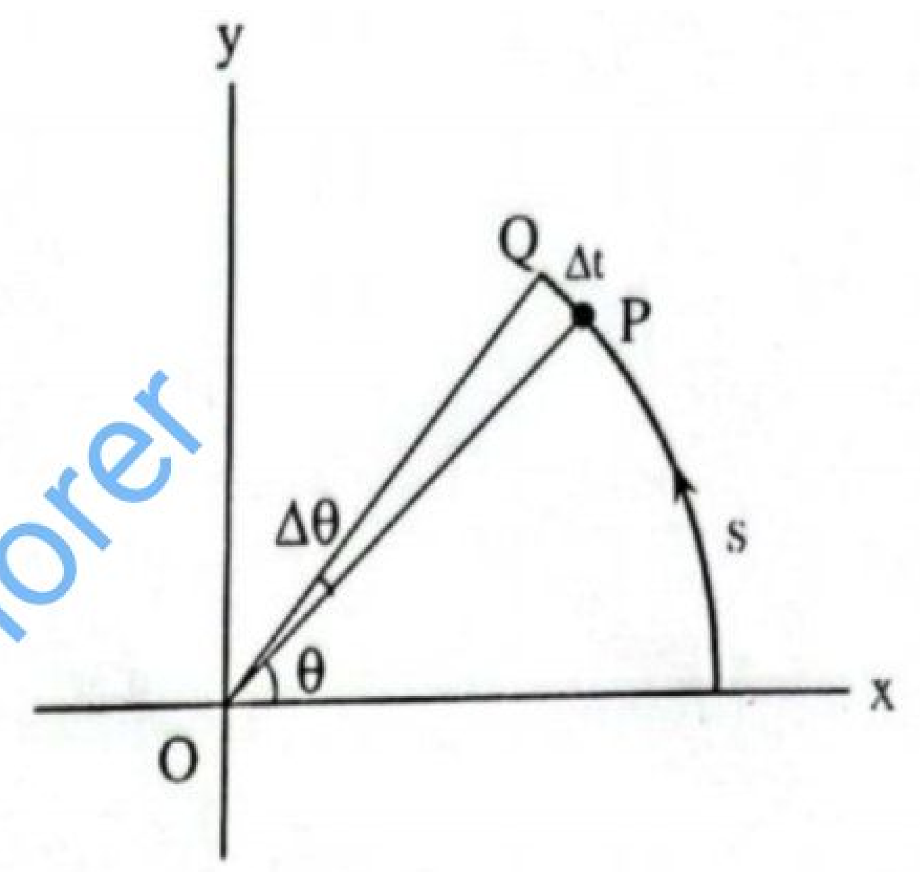


Fig. 2 Angular displacement of the rigid body is θ

When a rigid body rotates about an axis, its position is described by an angular displacement θ. Each particle of the body has a linear displacement s, the relation between the two displacement is

$$\text{Angle} = \frac{\text{Arc}}{\text{Radius}}$$

$$\theta = \frac{s}{r}$$

$$\boxed{\therefore s = \theta r}$$

If the angular displacement is $d\theta$ in time interval dt, then angular velocity ω is given by,

$$\omega = \frac{d\theta}{dt} \quad \text{——— eqn. ①}$$

If the angular velocity is not constant, the angular acceleration α is given by,

$$\alpha = \frac{d\omega}{dt} \text{ ——— eqn. (2)}$$

The particle of the rigid body in motion has linear velocity v which is tangent along the circular path, so

$$v = \frac{ds}{dt} = \frac{d(\theta r)}{dt} = r\frac{d\theta}{dt} = r\omega$$

$$\boxed{\therefore \; v = \omega r} \text{ ——— eqn. (3)}$$

The angular acceleration and linear acceleration are related as,

$$a = \frac{dv}{dt} = \frac{d\omega r}{dt} = r\frac{d\omega}{dt} = r\alpha$$

$$\boxed{\therefore a = \alpha r} \text{ ——— eqn. (4)}$$

Relation between linear and angular kinetics

Quantity	Relation
Displacement : θ and s	$s = \theta r$
Velocity : ω and v	$v = \omega r$
Acceleration: α and a	$a = \alpha r$

Equation of linear motion	Equation of Rotational motion
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$

★★★

$$s = \theta r$$

$$v = \omega r$$

$$a = \alpha r$$

Kinetic Energy of Rotation of Rigid Body

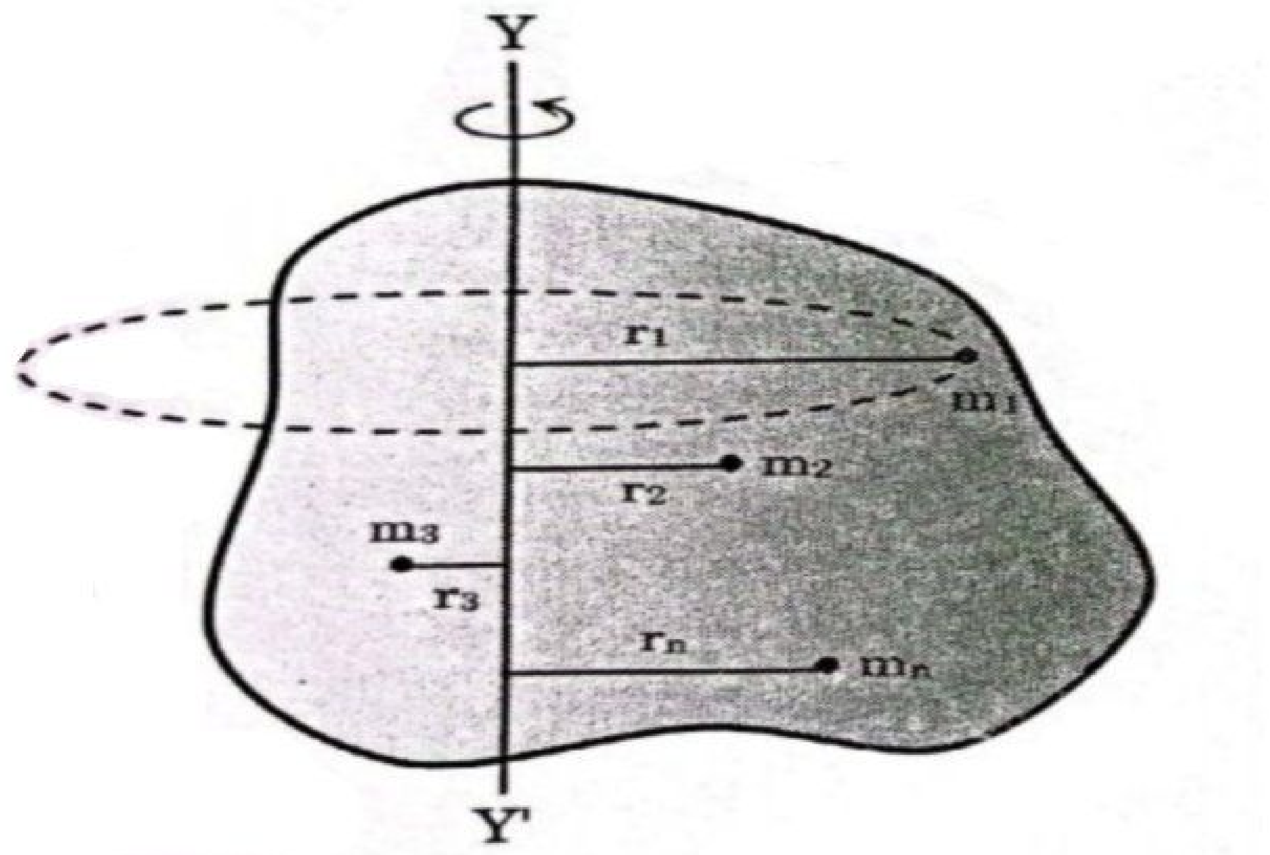


Fig. kinetic energy of rotating body

Consider a rotating rigid body of mass 'M' rotating about an axis YY' with an angular velocity 'ω'. Let the body be made up of large number of particles of masses $m_1, m_2, m_3 \ldots m_n$ which are situated at distances $r_1, r_2, r_3, \ldots r_n$ respectively from axis YY'. Although each particle within the body has same angular velocity ω, their linear velocity will be different.

Let $v_1, v_2, v_3 \ldots v_n$ be the linear velocities of the particles of masses $m_1, m_2, m_3 \ldots m_n$ respectively. Then

Rotational K.E of particle of mass $m_1 = \frac{1}{2} m_1 v_1^2$

$$= \frac{1}{2} m_1 (\omega r_1)^2 \quad [\because v = \omega r]$$

$$= \frac{1}{2} m_1 r_1^2 \omega^2$$

Now,

Rotational K.E of particle of mass $m_2, m_3, \ldots m_n$ will be $\frac{1}{2} m_2 r_2^2 \omega^2$, $\frac{1}{2} m_3 r_3^2 \omega^2, \ldots \frac{1}{2} m_n r_n^2 \omega^2$ respectively.

So,

Rotational K.E of body $= \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \frac{1}{2} m_3 r_3^2 \omega^2 + \ldots \frac{1}{2} m_n r_n^2 \omega^2$

$$= \frac{1}{2} \omega^2 (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots m_n r_n^2)$$

$$= \frac{1}{2} \omega^2 \left(\sum_{i=1}^{i=n} m_i r_i^2 \right)$$

$$= \frac{1}{2} \omega^2 I \quad \left[\because I = \sum m_i r_i^2 \right]$$

$$\boxed{\therefore \text{Rotational K.E} = \frac{1}{2} I \omega^2}$$

So, the rotational K.E of a body is equal to half the product of moment of inertia and the square of the angular velocity of the body about the axis.

Kinetic Energy of Rolling Body

↳ Linear Motion + Rotational motion

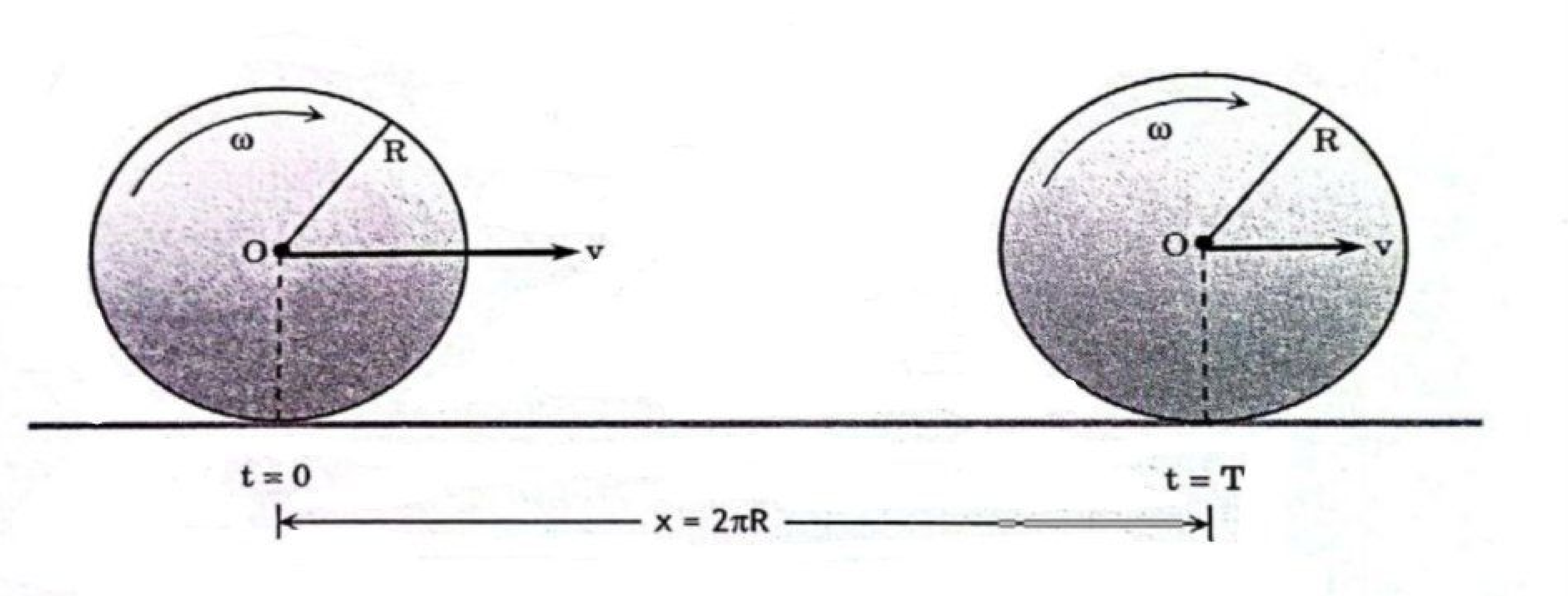


Fig. A rolling body on a flat surface

Consider a body such as wheel of mass 'm' and radius 'R' rolling along straight line on a horizontal plane surface. When the body rolls, it rotates about the horizontal axis and undergoes displacement in forward direction. So the body possesses both rotational and translational motion.

Let 'v' be the velocity of body and 'T' is the period of rotation of the body. During this time it covers an angle of 2π radian about the axis. If the body covers a linear distance 'x' in one revolution, then the distance covered in one revolution is

$$x = 2\pi R$$

The angular velocity of body $(\omega) = \frac{2\pi}{T}$ ——— eqn. ①

Linear velocity $(v) = \frac{2\pi R}{T}$ ——— eqn. ②

From eqn. ① and eqn. ②

$$v = \omega R$$

K.E of rotation $(K.E_{rot}) = \frac{1}{2} I\omega^2$

K.E of translation $(K.E_{trans}) = \frac{1}{2} mv^2$

∴ Total K.E of rolling body $= K.E_{rot} + K.E_{trans}$

$$= \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2$$

We have,

$$I = mK^2$$

where, K = Radius of gyration

Now,

$$\text{Total K.E} = \frac{1}{2}mK^2\omega^2 + \frac{1}{2}mv^2$$

$$= \frac{1}{2}mK^2\left(\frac{v}{R}\right)^2 + \frac{1}{2}mv^2$$

$$= \frac{1}{2}mv^2\left(\frac{K^2}{R^2} + 1\right)$$

$$\boxed{\therefore \text{Total K.E} = \frac{1}{2}mv^2\left(1 + \frac{K^2}{R^2}\right)}$$

Acceleration of a Rolling Body on a Inclined Plane

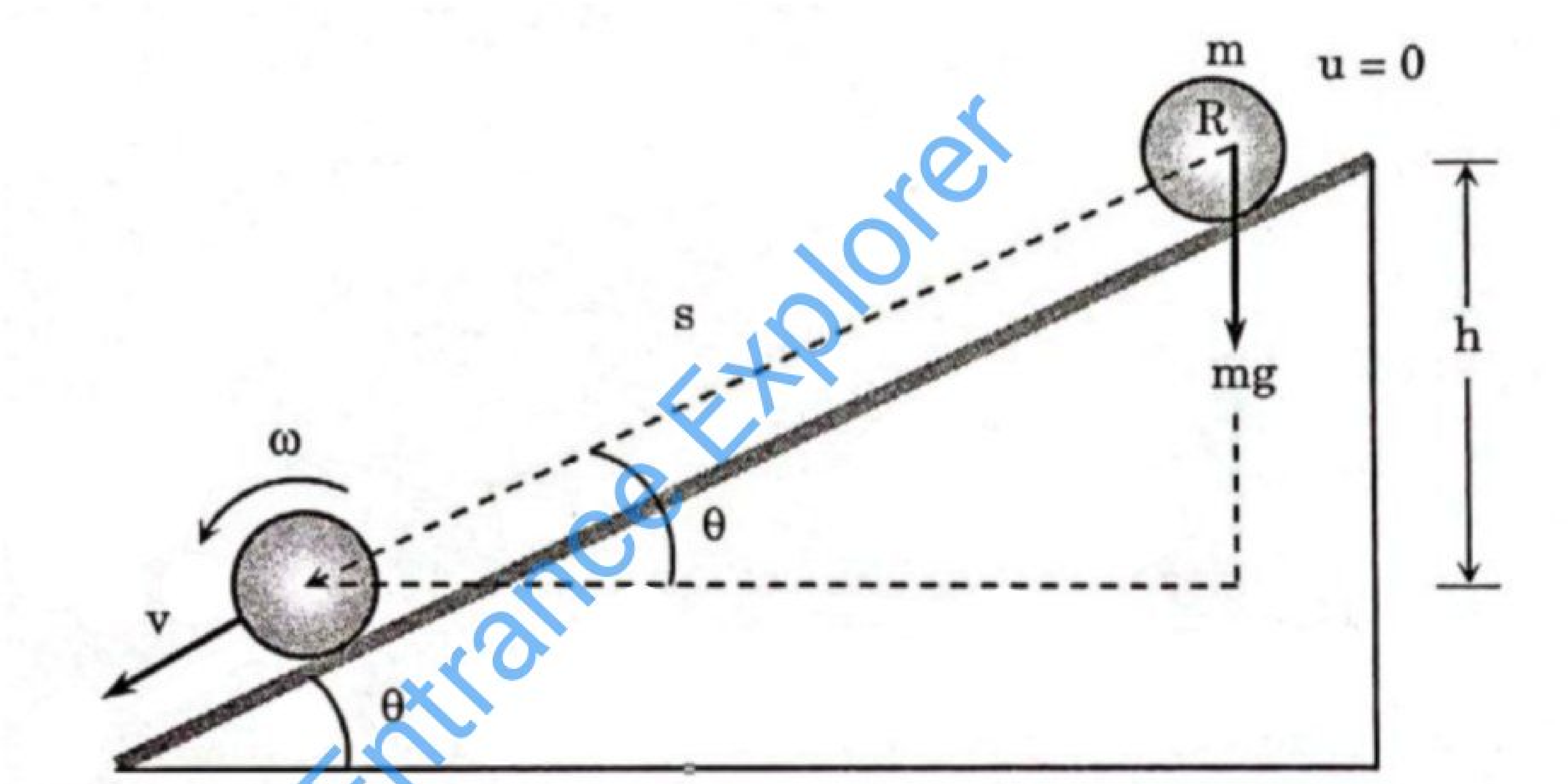


Fig. A body rolling on an inclined plane

Consider a circular body of mass 'm' and radius 'R' rolling down along a plane inclined to the horizontal at an angle θ.

If 'v' be the linear velocity acquired by the body on covering a distance 's' along the plane, it descends through a vertical height 'h' and loses potential energy.

Potential energy lost by the body = mgh

This must be equal to K.E gained by the body which is given by

$$\text{Total K.E gained by the body} = \frac{1}{2}mv^2\left(1 + \frac{K^2}{R^2}\right)$$

As, no slipping occurs, mechanical energy is conserved. So,

Loss in P.E = Gain in K.E

$$\text{or, } \cancel{m}gh = \frac{1}{2}\cancel{m}v^2\left(1+K^2/R^2\right)$$

$$\text{or, } v^2 = \frac{2gh}{1+K^2/R^2} \text{ ——— eqn (1)}$$

Also we have, In Δ

$$\sin\theta = \frac{h}{s}$$

$$\text{or, } h = s\sin\theta$$

Putting the value of h in eqn (1)

$$v^2 = \frac{2gs\sin\theta}{\left(1+K^2/R^2\right)} \text{ ——— eqn (2)}$$

Hence, initial velocity (u) = 0 for a body starting from rest, the eqn. of motion for the body will be

$v^2 = 2as$ where, a = acceleration of rolling body

$$\text{or, } \frac{\cancel{2}g\cancel{s}\sin\theta}{\left(1+K^2/R^2\right)} = \cancel{2}a\cancel{s}$$

$$\text{or, } \boxed{a = \frac{g\sin\theta}{\left(1+K^2/R^2\right)}} \text{ ——— eqn (3)}$$

eqn (3) is general expression for acceleration of body rolling down an inclined plane.

$$a = \alpha R$$

α ↳ Angular acceleration

Moment of Inertia

A body rotating about an axis opposes any change desired to be produced in its state. This property is known as moment of inertia.

Denoted by I.

$$I = mr^2$$

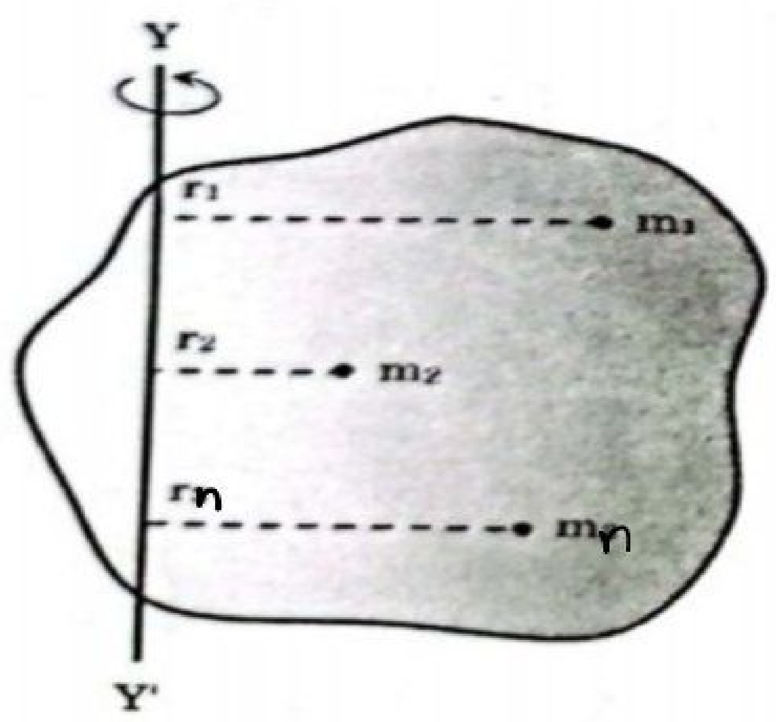


Fig. A rigid body rotating about an axis YY'

Consider a rigid body consisting a large number of small particles of m_1, m_2, ---m_n. Suppose the body be rotating about an axis YY' and the distance of these particles from this axis is r_1, r_2, ---r_n. The moment of inertia of these particles about the axis of rotation YY' will be $m_1r_1^2, m_2r_2^2$, --$m_nr_n^2$ respectively.

The moment of inertia 'I' of the body about the axis of rotation YY' is the sum of the moment of inertia of all particles about the axis YY' of which the body is made.

$$\therefore I = m_1r_1^2 + m_2r_2^2 + --- m_nr_n^2$$
$$= \sum_{i=1}^{n} m_ir_i^2$$

where, m_i = mass of ith particle

r_i = Radius of particle or distance of particle from the axis of rotation

$\therefore$ Moment of inertia is the sum of the product of the masses of various particles and square of their perpendicular distance from the axis of rotation for rigid body about given axis of rotation.

Unit of moment of inertia (I) = $kg\,m^2$ (SI unit)
= $g\,cm^2$ (CGS unit)

Dimensional formula = $[ML^2T^0]$

Moment of inertia varies with axis of rotation chosen and distribution of mass of the body.

Radius of Gyration

It is defined as the distance from the axis of rotation to a point where total mass of the body is supposed to be concentrated, so the moment of inertia about the axis may remain the same.

It is denoted by K.

$$\boxed{I = MK^2} \text{ ——— eq}^n \text{ (1)}$$

Suppose a body having 'n' number of particles each of mass m. Let r_1, r_2, ----r_n be their perpendicular distances from the axis of rotation. Then the moment of inertia 'I' of body about the axis of rotation is

$$I = mr_1^2 + mr_2^2 + --- mr_n^2$$

$$\text{or, } I = m(r_1^2 + r_2^2 + --- r_n^2)$$

$$\text{or, } I = \frac{mn}{n}(r_1^2 + r_2^2 + --- r_n^2)$$

where,

$$\text{or, } I = M\left(\frac{r_1^2 + r_2^2 + --- r_n^2}{n}\right)$$

$mn = M$ (Total mass of body)

$$\text{or, } \cancel{M}k^2 = \cancel{M}\left(\frac{r_1^2 + r_2^2 + --- r_n^2}{n}\right) \quad [\text{from eq}^n. 1]$$

$$\boxed{\therefore k = \sqrt{\frac{r_1^2 + r_2^2 + --- r_n^2}{n}}} \quad \text{——— eq}^n. ②$$

So, radius of gyration of body about a given axis may also be defined as root mean square distance of the various particles of the body from the axis of rotation.

Theorem of Parallel and Perpendicular Axis

Theorem of Parallel Axis

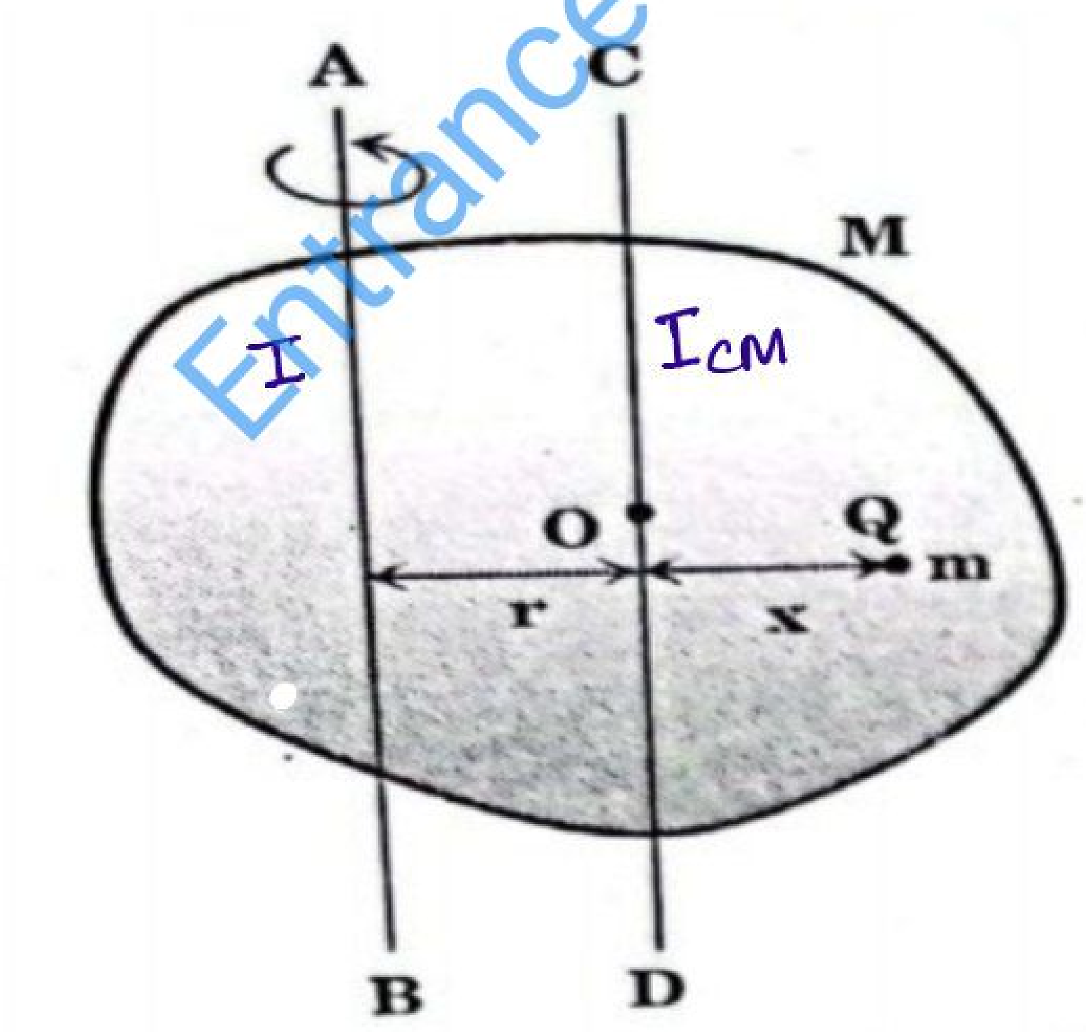


Fig. Theorem of parallel axis

"The moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its center of mass plus the product of the mass of body and the square of the distance between two axis."

From parallel axis theorem,

$$\boxed{I = I_{cm} + Mr^2}$$

Theorem of Perpendicular Axis

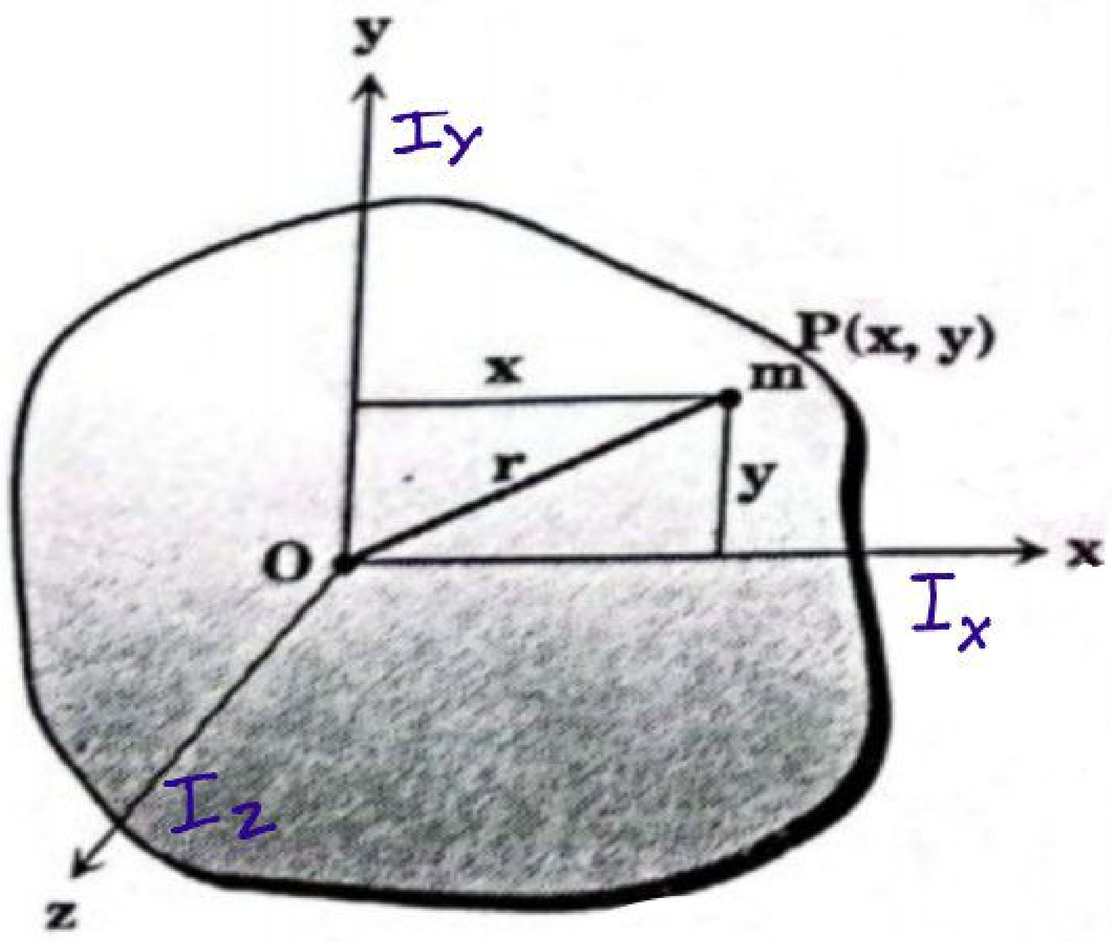


Fig. Theorem of perpendicular axis

"The sum of moment of inertia of a laminar body about any two mutually perpendicular axis in its plane is equal to its moment of inertia about an axis perpendicular to its plane and passing through the point of intersection of two axis"

From Theorem of perpendicular axis,

$$\boxed{I_z = I_x + I_y}$$

Calculation of Moment of Inertia of Rigid Bodies

1. Thin Uniform Rod

a) About an axis through its center and perpendicular to its length

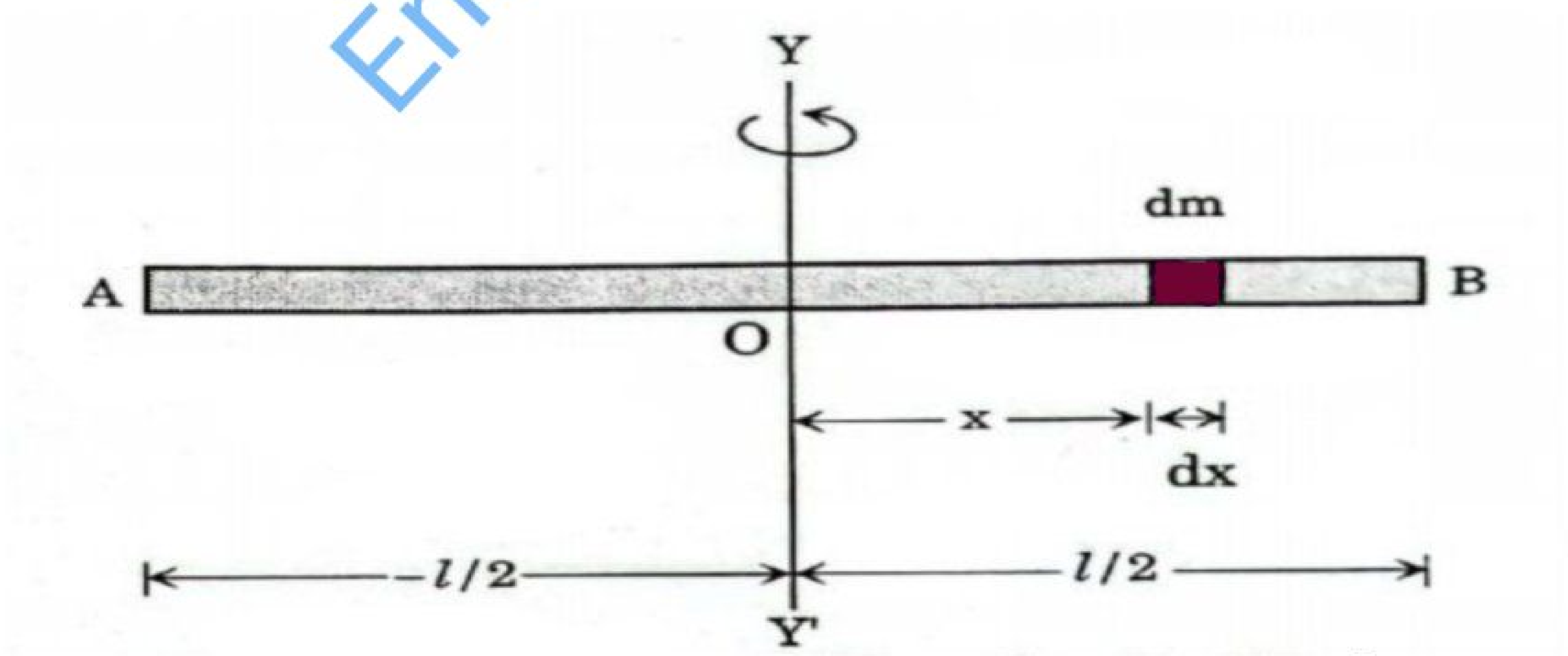


Fig. Moment of inertia of thin uniform rod

Consider a thin uniform rod of mass 'M' and length 'l'. Suppose the rod be rotating about an axis yy^1 passing through its center and perpendicular to its length. To find the moment of inertia of this rod about axis yy^1. Consider a small element of length 'dx' whose mass is 'dm' at a distance 'x' from the center O. Then,

Moment of inertia of this small element $= x^2 dm$

So, Moment of inertia of the rod about yy^1 is

$$I = \int_{-l/2}^{l/2} x^2 \, dm$$

Total mass of rod $= M$

Mass per unit length $= M/l$

Mass of element of length $dx = \frac{M}{l} dx$

or, $dm = \frac{M}{l} dx$

Now,

$$I = \int_{-l/2}^{l/2} x^2 \frac{M}{l} \, dx$$

$$= \frac{M}{l} \int_{-l/2}^{l/2} x^2 \, dx \qquad \left[\because \int x^n dx = \frac{x^{n+1}}{n+1} \right]$$

$$= \frac{M}{l} \left[\frac{x^3}{3} \right]_{-l/2}^{l/2}$$

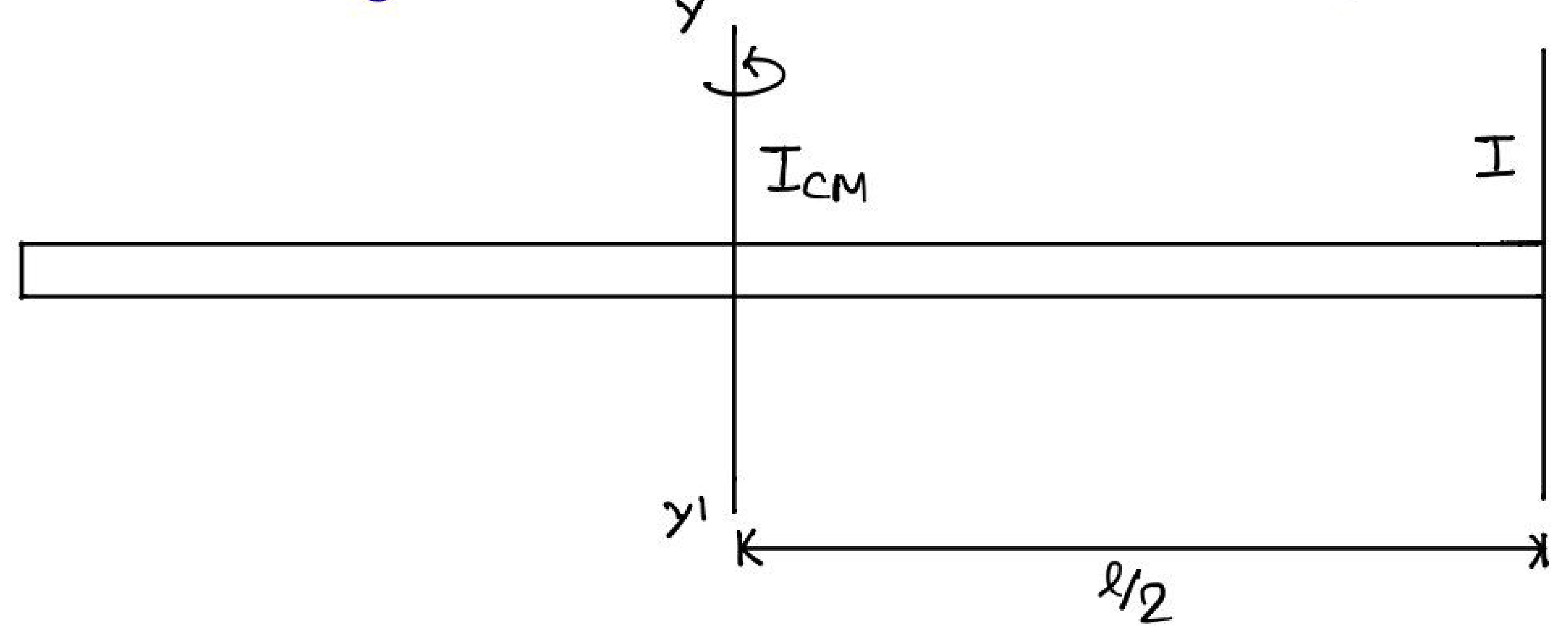
$$= \frac{M}{3l} \left[(l/2)^3 - (-l/2)^3 \right]$$

$$= \frac{M}{3l} \left[\frac{l^3}{8} + \frac{l^3}{8} \right]$$

$$= \frac{M}{3l} \times \frac{2l^3}{8} = \frac{Ml^2}{12}$$

$\boxed{\therefore I = \frac{Ml^2}{12}}$ → Required expression for the moment of inertia of thin uniform rod about an axis through its center and $\perp^r$ to its length

b> About an axis through one end and $\perp^r$ to its length



According to theorem of parallel axis, moment of inertia of the rod about an axis at the end of rod and perpendicular to it is

$$I = I_{CM} + M(l/2)^2$$

$$= Ml^2/12 + Ml^2/4$$

$$= \frac{Ml^2 + 3ML^2}{12}$$

$$= \frac{\cancel{4}ML^2}{\cancel{12}\,3}$$

$\boxed{\therefore I = \frac{ML^2}{3}}$ → Required expression for the moment of inertia of thin uniform rod about an axis through one end and $\perp^r$ to its length

2. Circular Ring

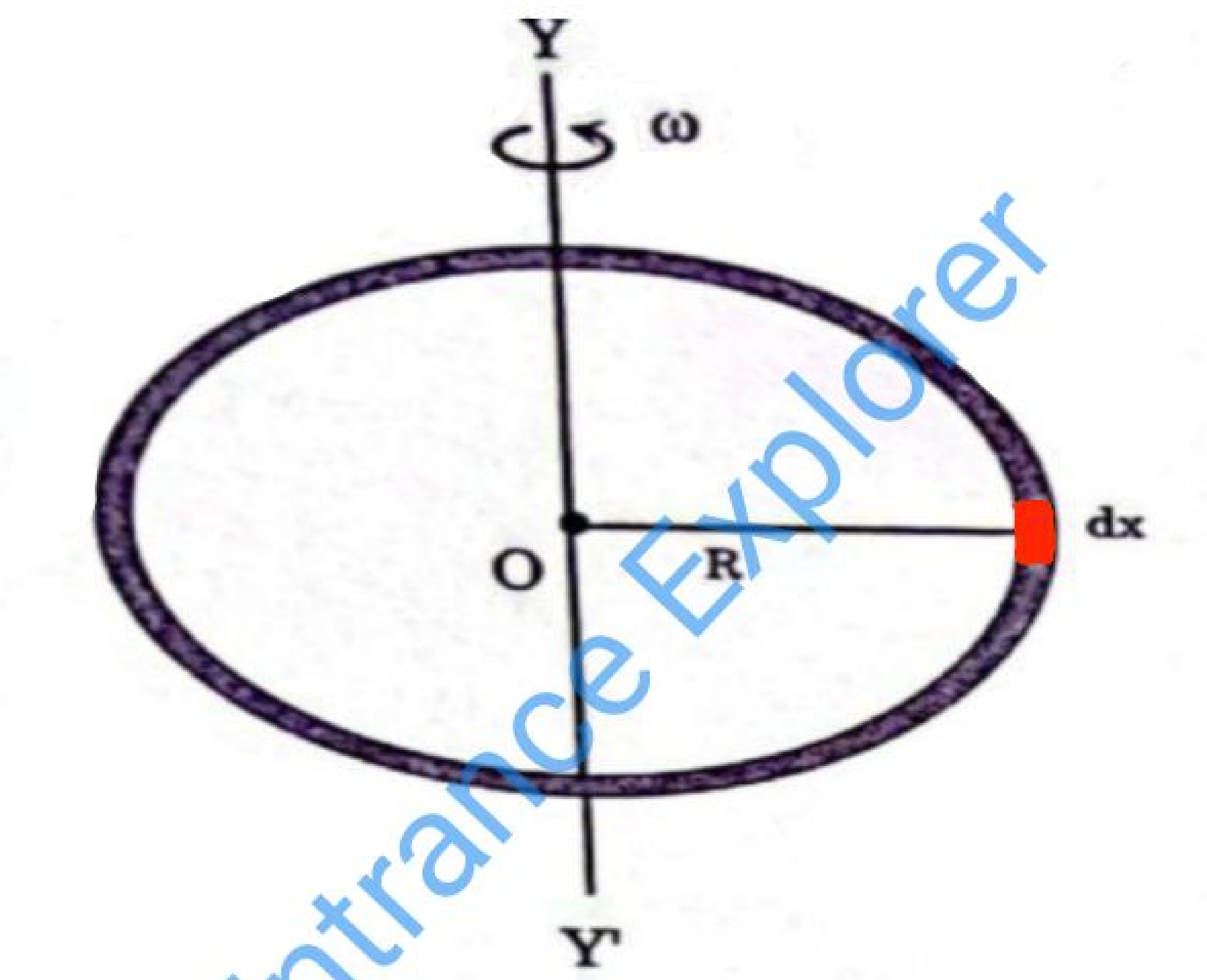


Fig. Moment of inertia of circular ring

Consider a thin uniform ring of mass 'M' and radius 'R'. Suppose the ring rotates about an axis yy' passing through its center and perpendicular to its plane.

Circumference of ring $= 2\pi R$

Total mass $= M$

Mass per unit length $= \frac{M}{2\pi R}$

Take a small element of length 'dx' in the ring which is at distance R from the axis.

Mass of length $dx = \frac{M}{2\pi R} dx$

$$\therefore dm = \frac{M}{2\pi R} dx$$

$\therefore$ Moment of inertia of the small element $= dm R^2$

$$= \frac{M}{2\pi \cancel{R}} dx . R^{\cancel{2}}$$

$$= \frac{MR}{2\pi} dx$$

Now, for moment of inertia I for whole ring,

$$I = \int_0^{2\pi R} \frac{MR}{2\pi} dx$$

$$= \frac{MR}{2\pi} \int_0^{2\pi R} dx$$

$$= \frac{MR}{2\pi} [x]_0^{2\pi R}$$

$$= \frac{MR}{2\pi} \times [2\pi R - 0]$$

$$= \frac{MR}{\cancel{2\pi}} \times \cancel{2\pi} R$$

$$= MR^2$$

$\boxed{\therefore I = MR^2}$ → Required moment of inertia

Torque and Angular Acceleration for a Rigid Body

Torque - Turning effect of a force in a body is called torque or moment of force

Torque = Force × Perpendicular distance of the force from axis of rotation

$\boxed{\therefore \tau = Fr \Rightarrow \vec{\tau} = \vec{r} \times \vec{F}}$ → Vector Quantity

SI unit → Nm

CGS unit → Dyne cm

Dimensional formula → $[MLT^{-2}][L] = [ML^2T^{-2}]$

Relation between torque and moment of inertia/angular acceleration

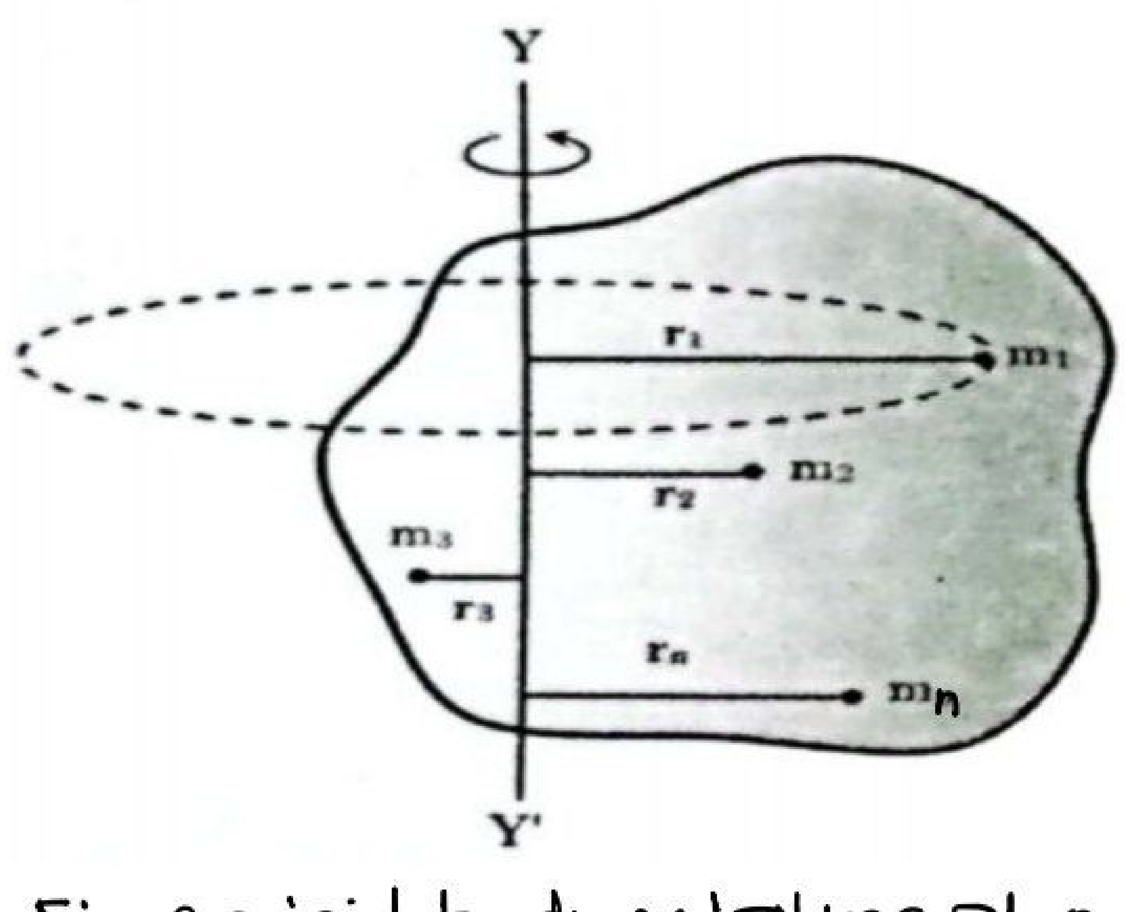


Fig. A rigid body rotating about an axis yy'

Consider a body rotating about an axis yy' under the action of constant torque (τ). Suppose the body consists of 'n' particles of masses m_1, m_2 ... m_n at distances of $r_1, r_2 .. r_n$ respectively.

The torque will produce a constant angular acceleration 'α' in each particle. Since the particle of mass m_1 follows a circular path of radius r_1, the magnitude of linear acceleration of this particle is,

$$a_1 = r_1\alpha$$

The net external force acting on it,

$$F_1 = m_1 a_1 = m_1 r_1 \alpha$$

The magnitude of torque acting upon this particle due to force,

$$\tau_1 = F_1 r_1$$
$$= (m_1 r_1 \alpha) r_1$$
$$= m_1 r_1^2 \alpha$$

Similarly, the magnitude of torque on the particle of masses $m_2, \ldots m_n$ are $m_2 r_2^2\alpha, \ldots m_n r_n^2 \alpha$ respectively.

$$\therefore \text{Torque on the body } (\tau) = m_1 r_1^2\alpha + m_2 r_2^2 \alpha + \ldots m_n r_n^2 \alpha$$
$$= (m_1 r_1^2 + m_2 r_2^2 + \ldots m_n r_n^2)\alpha$$
$$= \left(\sum_{i=1}^{n} m_i r_i^2\right)\alpha$$

$$\boxed{\therefore \tau = I\alpha}$$

→ Required relation between moment of inertia angular accn. and torque of body

If $\alpha = 1$ then $\tau = I$

Hence, the moment of inertia of body about given axis is equal to the torque required to produce unit angular accn. in the body.

Work and Power in Rotational Motion

Two equal and opposite parallel forces acting on a rigid body at different points such that their line of action do not coincide constitute a couple.

Consider a wheel of radius 'r' rotating about its center O. Suppose two equal and opposite forces 'F' act tangentially at points A and B. Let 'θ' be the angle of rotation of the wheel. The torque due to two forces is constant.

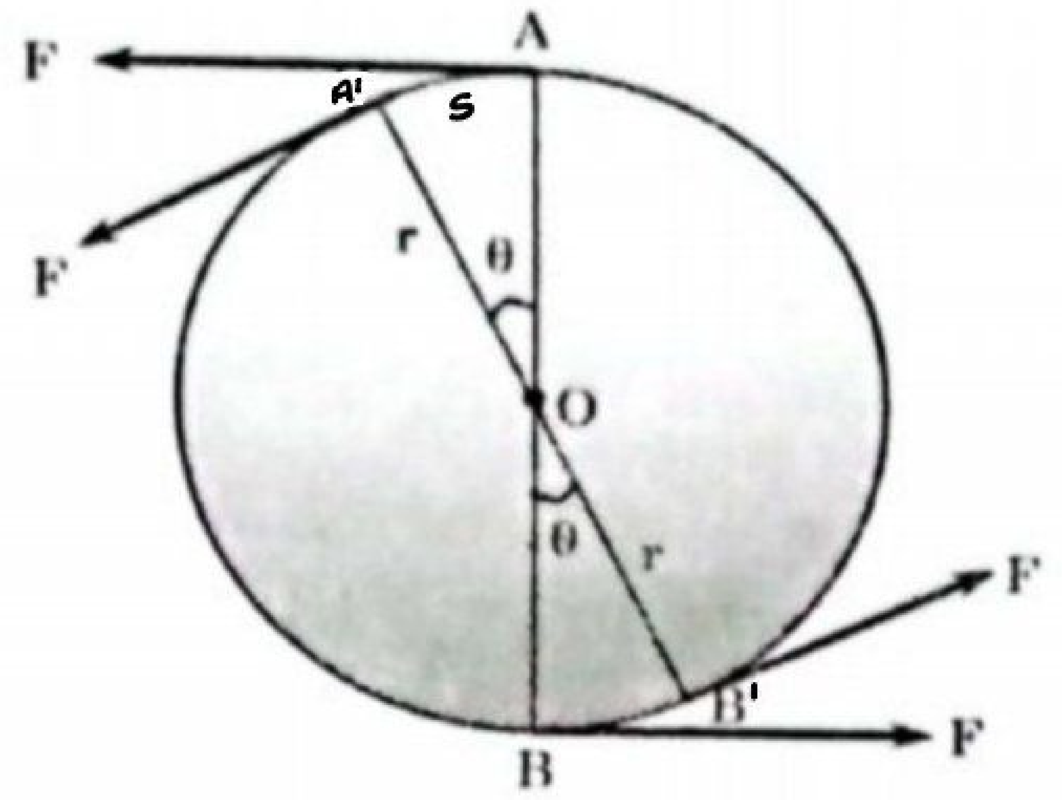


Fig. Work done by couple in rotating wheel

Work done by each force = Force × Distance

$= F \times S$

$= F \times r\theta \quad [\because S = r\theta]$

$= Fr\theta$

Where θ is in radian and distance $= AA' = BB' = S = \theta r$

$\therefore$ Total work done $(W) = Fr\theta + Fr\theta$

$= 2Fr\theta$ ______ eqn. ①

But, Torque $(\tau) = F \times 2r$

$= 2Fr$ ______ eqn. ②

From eqn. ① and eqn. ②

$\boxed{\therefore W = \tau\theta}$ → Work done by a couple is the product of torque and angle of rotation

Now,

$$\text{Power}(P) = \frac{dW}{dt} = \frac{d(\tau\theta)}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$$

$\boxed{\therefore P = \tau\omega}$ → Power is the product of torque and angular velocity

Angular Momentum

Moment of linear momentum of an object is called angular momentum. It is measured by the product of the linear momentum of the object and the perpendicular distance between the object and axis of rotation.

Angular momentum (L) = Linear momentum (P) × $\perp^r$ distance from axis of rotation (r)

$$\boxed{L = mvr}$$

$$L = mvr = m(\omega r)r = m\omega r^2 = (mr^2)\omega = I\omega$$

It is vector quantity.

SI unit - $kg\frac{m}{s}m = kgm^2s^{-1}$

CGS unit - gcm^2s^{-1}

Dimensional formula $= [ML^2T^{-1}]$

Relation between angular momentum and torque

We know,

$$L = I\omega$$

$$\text{or, } \frac{dL}{dt} = I\frac{d\omega}{dt}$$

$$\text{or, } \frac{dL}{dt} = I\alpha \quad (\because \alpha = \frac{d\omega}{dt})$$

$$\text{or, } \frac{dL}{dt} = \tau \quad (\because \tau = I\alpha)$$

$$\text{or, } \tau = \frac{dL}{dt}$$

$$\boxed{\therefore \tau = \frac{dL}{dt}}$$ → Torque is also rate of change of angular momentum of the body

Principle of conservation of angular momentum

"If no external torque acts on system, the total angular momentum of the system remains conserved."

If 'I' be the moment of inertia of a body about a given axis of rotation and 'ω' is its angular velocity, then

$$L = \text{Constant}$$

$$I\omega = \text{constant}$$

$$I_1\omega_1 = I_2\omega_2$$

Proof:

We have, $\frac{dL}{dt} = \tau$

If no external torque acts on the system, $\tau = 0$

$$\text{or, } \frac{dL}{dt} = 0$$

Integrating on both sides,

$$L = \text{Constant} \Rightarrow I\omega = \text{Constant} \Rightarrow I_1\omega_1 = I_2\omega_2$$

Examples:

1.> A diver uses the principle of conservation of angular momentum i.e

$$L = I\omega = \text{Constant}$$

While diving into a swimming pool from a diving boad, the diver cuts cuts his body by rolling his arms and legs. In this case, her moment of inertia decreases and angular velocity increases keeping the total angular momentum constant. So, she is able to make more somersaults before striking the water.

But before entering water surface, she stretches her arms and legs so that her moment of inertia increases and the angular velocity decreases and she is able to touch the water surface with a reduced speed without hurting.

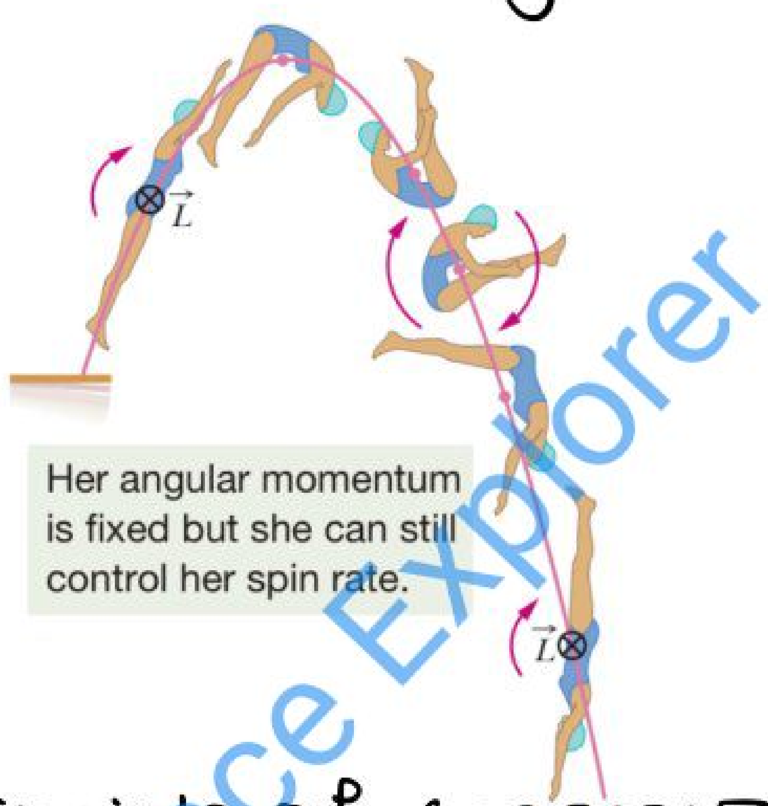


Fig. A diver applying principle of conservation of angular momentum

2.> A man standing on a rotating turn table holding weight on his hands, can rotate with desired angular velocity using the principle of conservation of angular momentum. When he outstretches his hands, the moment of inertia increases and the speed of rotation decreases so that angular momentum remains constant.

On the other hand, as he draws his hands to the chest, the momentum of inertia decreases and the angular velocity increases. Hence, the spinning rate of the man increases.

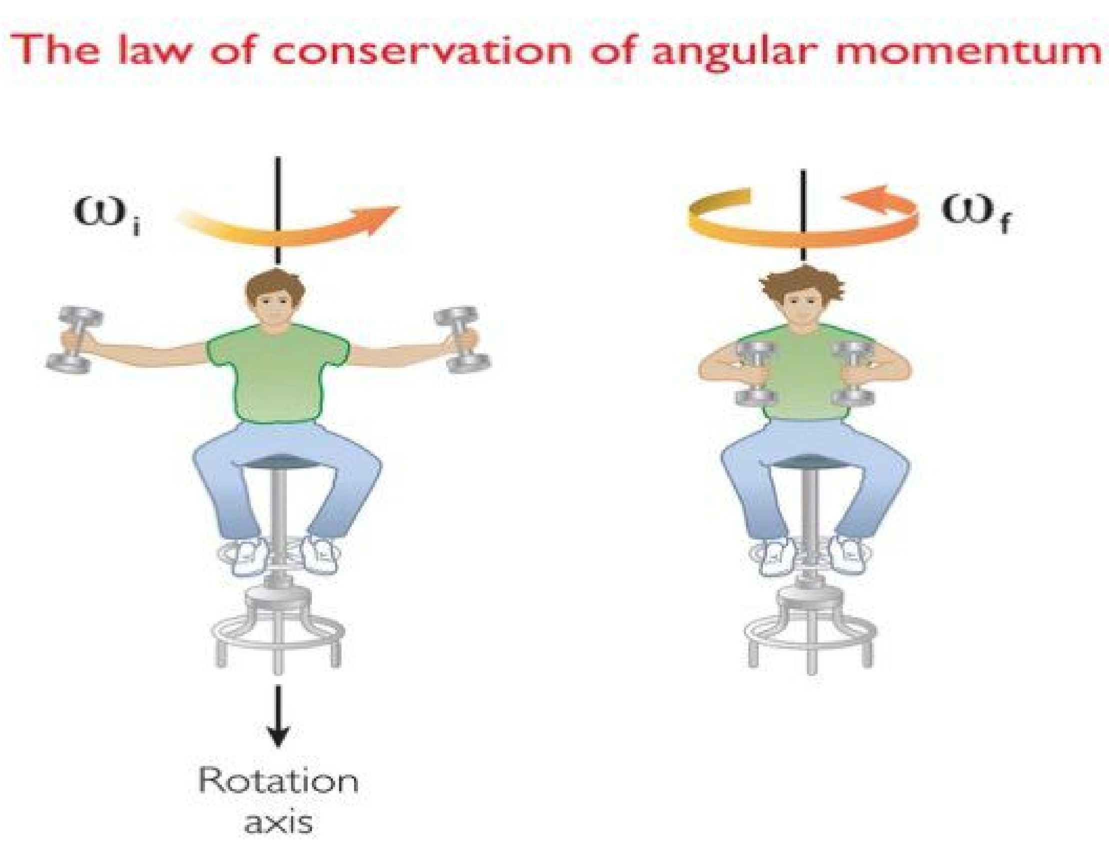


Fig. A man on rotating table