Periodic Motion

A motion which repeats after a fixed interval of time is called periodic motion.

Eg. Motion of planet around the sun
Motion of hour hand or minute hand of clock

Simple Harmonic Motion and Equations

If a particle moves to and fro about mean position in a straight line such that the acceleration is directed towards the mean position and is directly proportional to the displacement from that position is called simple harmonic motion.

Or

The harmonic motion of simplest type i.e constant amplitude and of single frequency is called SHM.

$$\text{Acceleration} \propto \text{Displacement}$$

$$\text{or, Acceleration} = -k \times \text{displacement}$$

where,

K= constant (-ve sign shows that acceleration is directed in opposite to the motion of object)

Characteristics of Simple Harmonic Motion

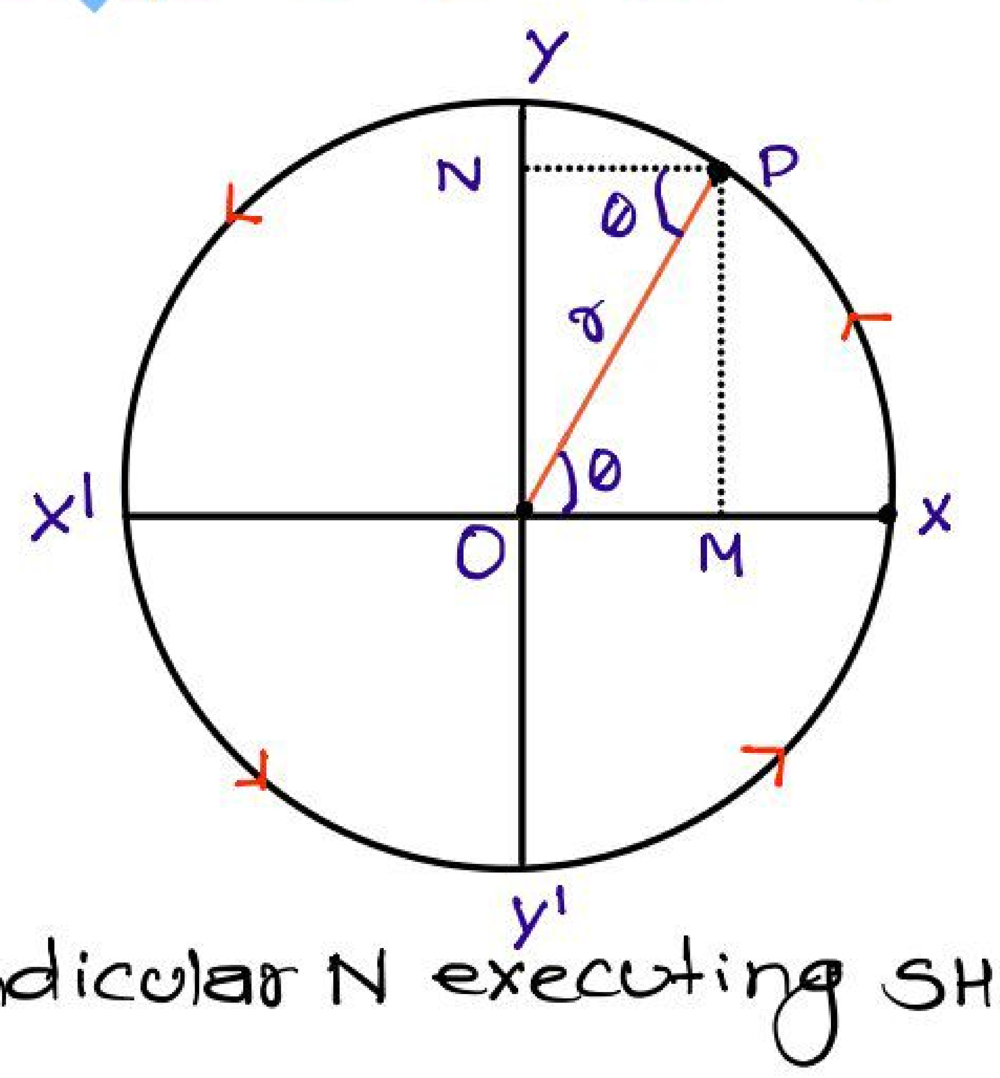

Fig. Foot of perpendicular N executing SHM on the diameter YOY'

Consider a particle moving around a circle of radius 'r' with a uniform angular velocity 'ω' in anticlockwise direction as shown in figure. XOX' and YOY' are two mutually perpendicular diameters of circle. As the particle goes in the circle the foot of perpendicular along Y-axis executes oscillatory motion.

Let at any time 't' the position of particle is 'P' and angular displacement θ. Let 'M' and 'N' be the foot of the perpendicular drawn from 'P' on XOX' and YOY' respectively. The displacement of the foot of perpendicular 'N' on diameter YOY' is ON.

In ΔONP,

$$\sin\theta = \frac{ON}{OP} = \frac{y}{r}$$

$$\boxed{\therefore y = r\sin\theta = r\sin\omega t} \text{ ——— eqn. 1}$$

This is displacement eqn. for SHM which is periodic. It can be expressed in terms of cosine function and similar expression can be obtained in any diameter of circle.

Velocity (v)

Rate of change of displacement w.r.t time. So,

$$v = \frac{dy}{dt} = \frac{d(r\sin\omega t)}{dt} = r\frac{d\sin\omega t}{dt}$$

$$\boxed{\therefore v = r\omega\cos\omega t} \text{ ——— eqn. 2}$$

$$\text{or, } v = r\omega\sqrt{1-\sin^2\omega t} \text{ ——— eqn. 3}$$

$$\text{or, } v = r\omega\sqrt{1-y^2/r^2}$$

$$\text{or, } v = r\omega\sqrt{\frac{r^2-y^2}{r^2}}$$

$$\text{or, } v = \frac{\cancel{r}\omega}{\cancel{r}}\sqrt{r^2-y^2}$$

$$\boxed{\therefore v = \omega\sqrt{r^2-y^2}} \text{ ——— eqn. 4}$$

This eqn. shows that velocity is not uniform

Case 1:

When $y=0$, $v_{max} = \omega r$ i.e at mean position, velocity is max.

Case 2:

When $y = r$, $v = 0$ i.e at extreme position, velocity is zero.

Acceleration (a)

Rate of change of velocity w.r.t time. So,

$$a = \frac{dv}{dt} = \frac{d(r\omega\cos\omega t)}{dt} = r\omega\frac{d\cos\omega t}{dt} = r\omega(-\omega\sin\omega t)$$

$$\boxed{\therefore a = -\omega^2 r \sin\omega t = -\omega^2 y}$$

Case 1:

When $y=0$, $a_{min}=0$, i.e at mean position, the accleration is zero.

Case 2:

when $y=r$, $a_{max}=-\omega^2 r$, i.e at extreme position, acceleration is maximum.

Amplitude (A)

The maximum displacement of the particle from the mean position is called amplitude.

We have, Displacement $(y) = r\sin\omega t$

when $\sin\omega t = 1$ then $y_{max} = r$ So,

Amplitude of motion $= r$

Time period (T)

Time taken by the particle to complete one oscillation in SHM is called time period.

We have,

$$a = \omega^2 y$$

$$\text{or, } \omega = \sqrt{a/y}$$

$$\text{or, } \frac{2\pi}{T} = \sqrt{a/y}$$

$$\text{or, } T = 2\pi\sqrt{y/a}$$

$$\boxed{\therefore T = 2\pi\sqrt{y/a}}$$

Frequency (f)

Number of oscillations completed in 1 second.

$$\boxed{f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{a}{y}}}$$

Phase

Phase of an oscillatory particle at any instant gives the position and the direction of motion of the particle with respect to its mean position

Graphical Representation of Displacement, Velocity and Acceleration in SHM

For particle executing SHM,

Displacement $(y) = r\sin\omega t$

Velocity $(v) = r\omega\cos\omega t = r\omega\sin(\omega t + \pi/2)$

Acceleration $(a) = -r\omega^2\sin\omega t$

Value of t	value of y	value of v	value of a
$t=0$	0	$r\omega$	0
$t=T/4$	r	0	$-r\omega^2$
$t=T/2$	0	$-r\omega$	0
$t=3T/4$	$-r$	0	$r\omega^2$
$t=T$	0	$r\omega$	0

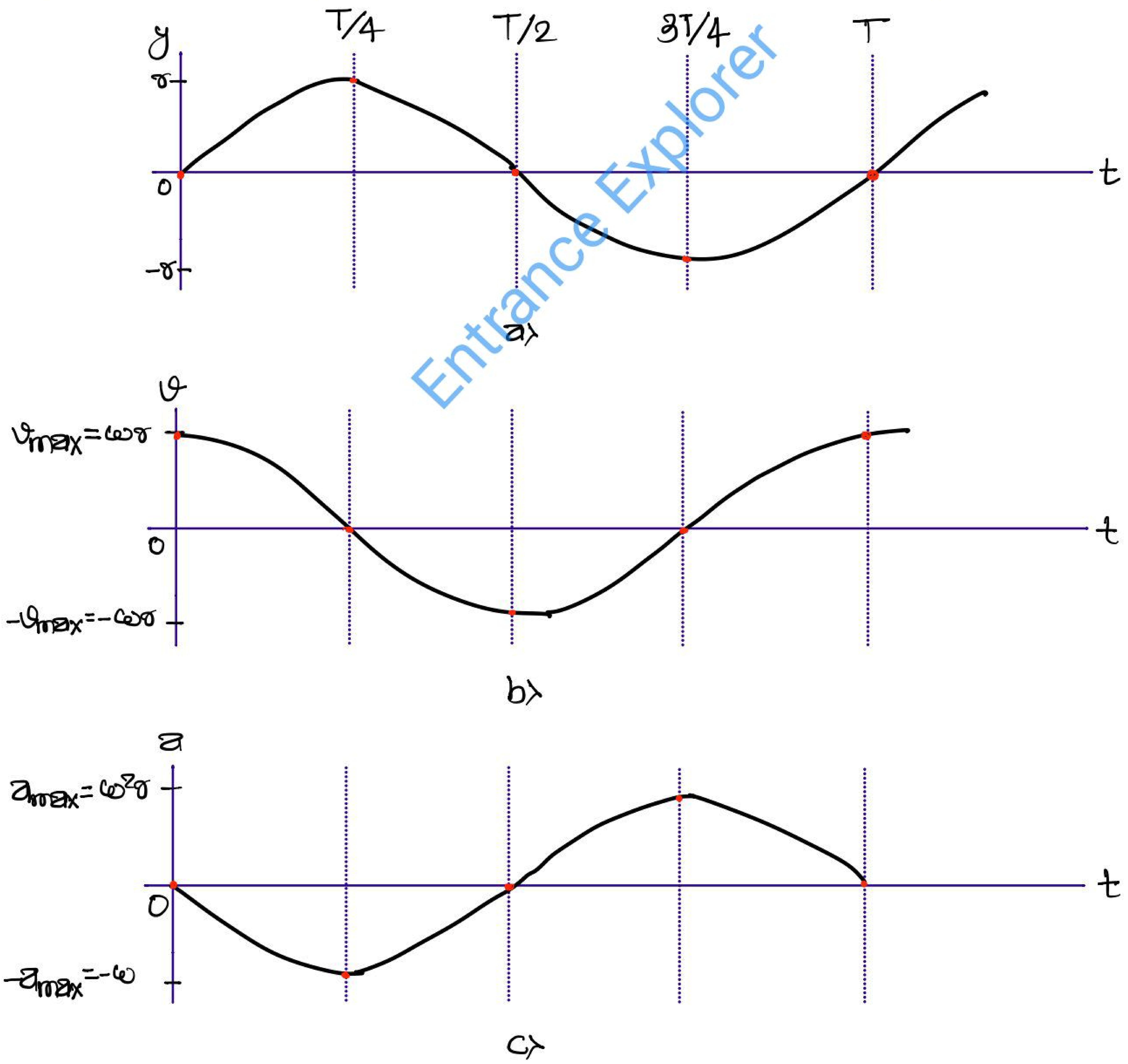

Fig. Graphical representation of a) displacement b) velocity and c) acceleration in SHM

Energy in SHM

Particle executing SHM has both the potential energy (PE) and kinetic energy (K.E)

If there is no damping in the oscillation, then total energy of the particle executing SHM is the sum of K.E and P.E. Since position and velocity of oscillating particle are changing continuosly with time, these energies changes continuosly but total energy is constant at every point.

Suppose a particle of mass 'm' is excecuting SHM with amplitude 'r' and angular velocity 'ω'. If 'y' is the displacement then

$$\text{Acceleration } (a) = -\omega^2 y$$

$$\text{Restoring force } (F) = ma$$

$$= -m\omega^2 y$$

$$= -ky$$

where, $k = m\omega^2$ is constant

If the particle is displaced further by a small displacement 'dy' against the force, then

$$\text{Work done } (dw) = -F dy$$

$$= -(-ky)dy$$

$$= ky\,dy$$

So, total work done to displace the particle from mean position to the position of displacement 'y' is

$$w = \int_0^y ky\,dy = k\left[y^2/2\right]_0^y = k\left(y^2/2 - 0\right) = \tfrac{1}{2}ky^2$$

$$\boxed{\therefore w = \tfrac{1}{2}ky^2}$$

The work done on the particle will remain in the form of potential energy. Thus

$$P.E = \tfrac{1}{2}ky^2$$

$$= \tfrac{1}{2}m\omega^2 y^2 \text{ ———— eq}^n \text{ (1)}$$

And, $K.E = \frac{1}{2}mv^2 = \frac{1}{2}m(\omega\sqrt{r^2-y^2})^2$

$= \frac{1}{2}m\omega^2(r^2-y^2)$ ———— eqn ②

Total energy of particle at any point (TE) = PE + KE

$= \frac{1}{2}m\omega^2y^2 + \frac{1}{2}m\omega^2(r^2-y^2)$

$= \frac{1}{2}m\omega^2(y^2+r^2-y^2)$

$= \frac{1}{2}m\omega^2r^2$

$= \frac{1}{2}m(2\pi f)^2r^2$

$= 2m\pi^2f^2r^2$

As m, v and r are constant, the total energy remains constant for a particle executing SHM.

Case 1:

When the particle is at mean position, $y=0$

Then, $PE = \frac{1}{2}m\omega^2y^2 = \frac{1}{2}m\omega^2(0)^2 = 0$

$KE = \frac{1}{2}m\omega^2(r^2-y^2) = \frac{1}{2}m\omega^2(r^2-0) = \frac{1}{2}m\omega^2r^2$

$TE = \underset{0}{\cancel{PE}} + KE = KE = \frac{1}{2}m\omega^2r^2$

Case 2:

When the particle is at extreme position, $y=r$

Then, $PE = \frac{1}{2}m\omega^2r^2$

$KE = 0$

$TE = PE + \overset{0}{\cancel{KE}} = PE = \frac{1}{2}m\omega^2r^2$

Application of SHM

a. Vibration of particle in Horizontal Spring

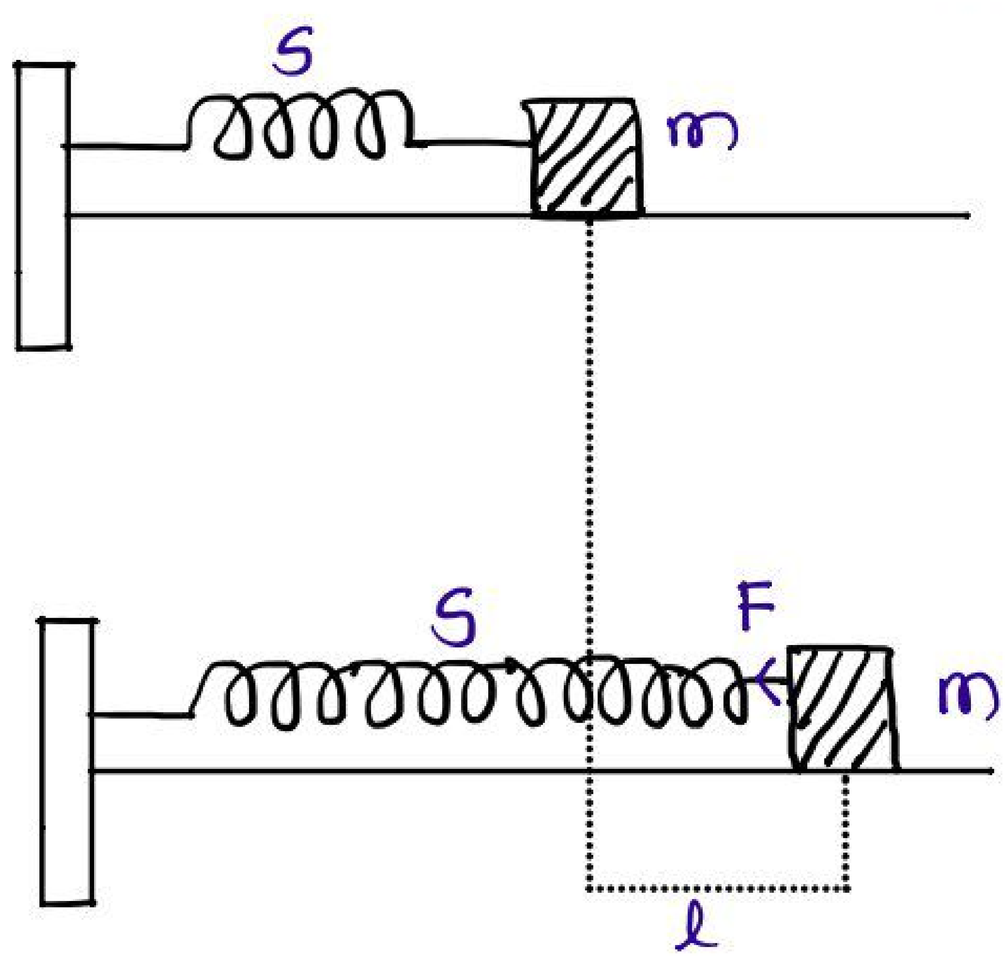

Fig. A mass oscillating horizontally on light stretched sring

Let, suppose that one end of spring 's' of negligible mass is attached to the wall and the other end to an object of mass 'm'. The spring 's' and object are laid on horizontal smooth table. If the mass is pulled slightly to extend the spring and then released, the system vibrates with SHM. 'O' is the position of mass at the end of spring corresponding to its natural length.

Let 'l' be the extension of the spring and 'F' be the restoring force set up in the spring. Then,

From Hooke's law,

$$F \propto l$$

$$\text{or, } F = -kl \text{ ——— eqn (1)}$$

where, k = force constant of spring or force per unit extension

-ve sign indicates that restoring force acts opposite to the displacement of mass

If 'a' is the acceleration produced in the mass, then

$$F = ma \text{ ——— eqn (2)}$$

From eqn (1) and eqn (2)

$$ma = -kl$$

$$\text{or, } a = -\frac{k}{m} l$$

$$\text{or, } a = -\omega^2 l \text{ ——— eqn (3)}$$

where, $\omega^2 = k/m$ is constant. This shows that acceleration is directly proportional to displacement and is directed towards the mean position.

Time period (T)

we have, $\omega^2 = k/m$

$$\text{or, } \left(\frac{2\pi}{T}\right)^2 = \frac{k}{m}$$

$$\text{or, } \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

$$\text{or, } T = 2\pi\sqrt{\frac{m}{k}}$$

$$\boxed{\therefore T = 2\pi\sqrt{\frac{m}{k}}}$$ — Required expression for time period of mass spring system

b. Vibration of particle in vertical spring

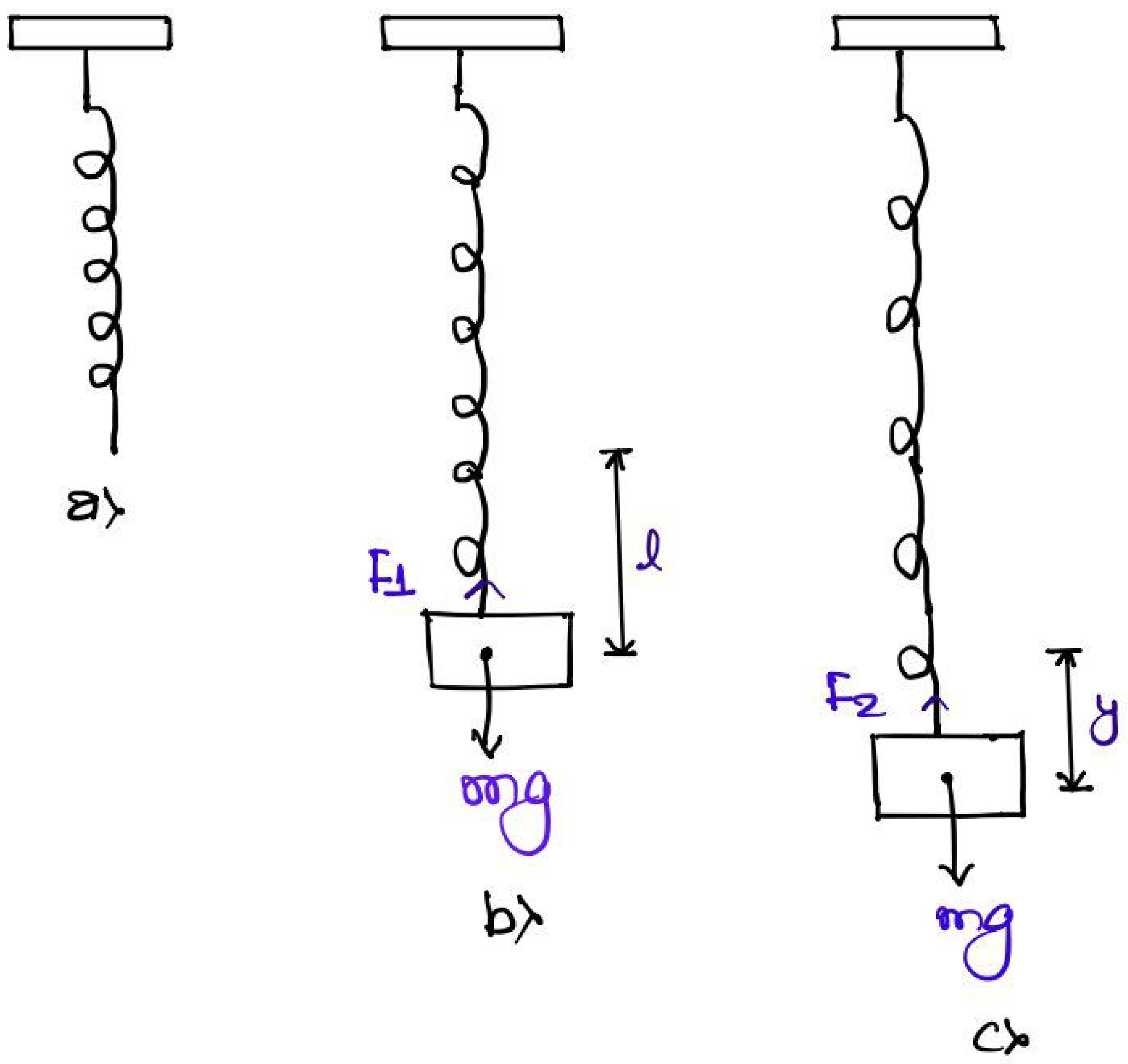

Fig. a> Spring attached to rigid support b> spring is stretched through 'l' when small mass is attached c> Vibration of mass as SHM

Consider elastic spring of force constant k. Let its one end be attached to a rigid support and a mass is attached at the other end.

When a load (m) is attached, the spring extends and let 'l' be the elongation produced. The restoring force on the spring is

$$F_1 = mg = -kl \text{ —— eq}^n \text{ (1)}$$

Let the load is pulled down through a small distance 'y' then the restoring force is

$$F_2 = -k(l+y) \text{ —— eq}^n \text{ (2)}$$

The effect of restoring force which cause oscillation is

$$F = F_2 - F_1$$

$$= -k(l+y) - (-kl)$$

$$= -\cancel{kl} - ky + \cancel{kl}$$

$$F = -ky$$

As, $F = ma$

$$\therefore ma = -ky$$

$$\text{or, } a = -k/m\, y$$

$$\text{or, } a = -\omega^2 y$$

$\boxed{\therefore a \propto y}$ Hence motion of vertical spring is SHM

Time period (T)

For SHM,

$$a = -\omega^2 y$$

For oscillating loaded spring,

$$a = -\frac{K}{m} y$$

or, $-\omega^2 \cancel{y} = -\frac{K}{m} \cancel{y}$

or, $\omega = \sqrt{\frac{K}{m}}$

or, $\frac{2\pi}{T} = \sqrt{\frac{K}{m}}$

$$\boxed{\therefore T = 2\pi \sqrt{\frac{m}{K}}}$$

Simple Pendulum

A simple pendulum is a heavy point mass object suspended by an inextensible, weightless and flexible string from a rigid support which is free to oscillate in a vertical plane.

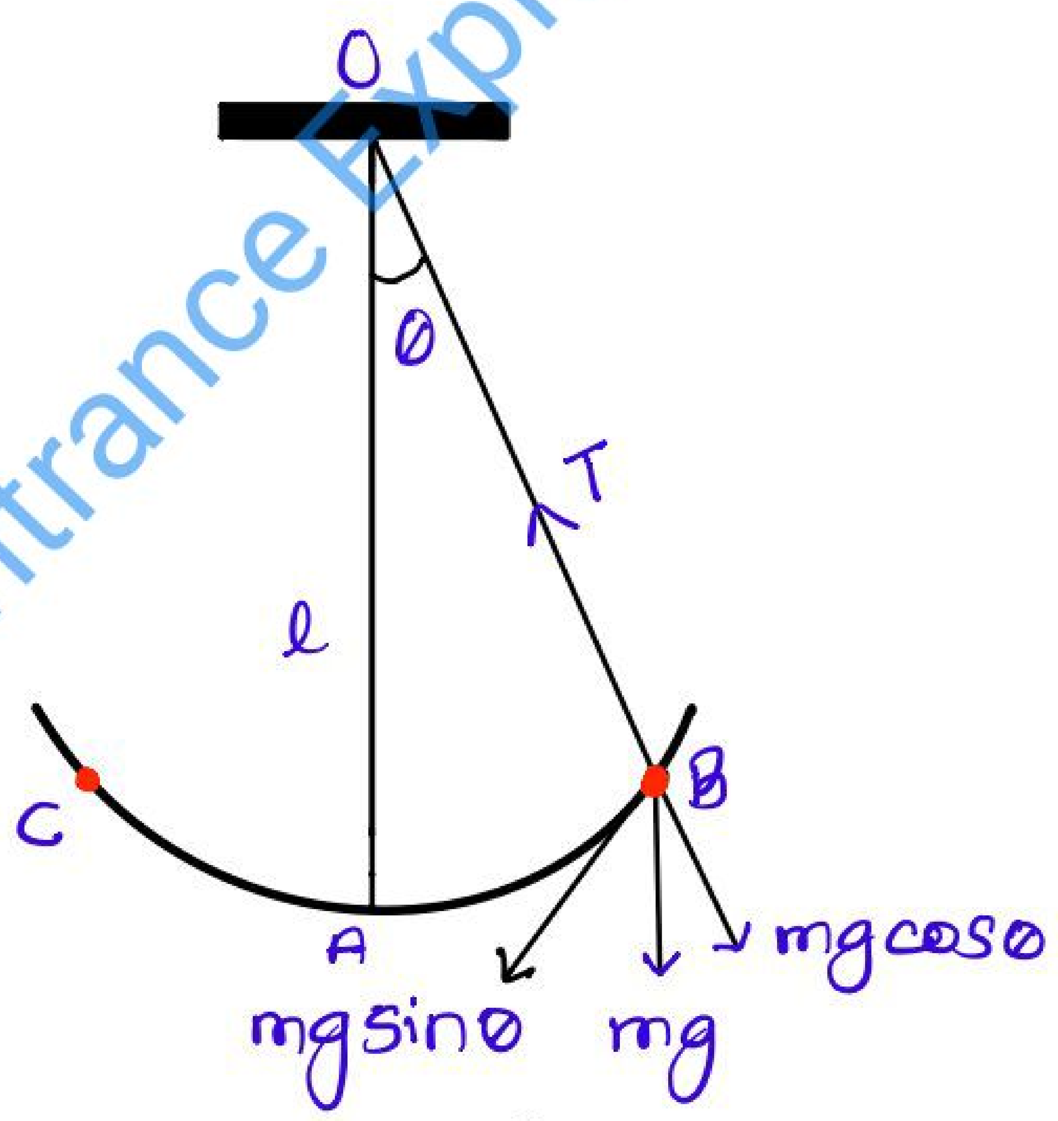

Fig. Simple Pendulum

Let 'm' be the mass of the bob of a simple pendulum whose length is 'l'. 'O' is the point of suspension and A is the mean position. When the bob is displaced from its mean position, it oscillates along path CAB in vertical plane and at any instant let angular displacement is θ.

The force acting on the bob at 'B' are

1. Weight mg of the bob acting vertically downward
2. Tension 'T' in the string along its length toward the point of suspension

The weight mg of the bob is resolved into two components $mg\cos\theta$ opposite to tension and $mg\sin\theta$ perpendicular to the string. The component $mg\cos\theta$ balances the tension in the string and $mg\sin\theta$ acts as restoring force and produces acceleration in the bob in the direction of arrow

$$F = -mg\sin\theta \quad \text{———— eqn. ①}$$

If a is the acceleration of the bob, then

$$F = ma = -mg\sin\theta$$

$$\therefore a = -g\sin\theta \quad \text{———— eqn. ②}$$

The (-ve) sign shows that the force is towards A, while the displacement 'y' is measured along the arc from A in opposite direction.

For small θ, $\sin\theta \sim \theta$

$$\theta = \frac{\text{Arc AB}}{l} = \frac{y}{l} \quad \text{———— eqn. ③}$$

From eqn. ② and eqn. ③

$$a = -g\theta = -g\, y/l$$

$$\boxed{\therefore a = -(g/l)\,y} \quad \text{———— eqn. ④}$$

Since, $\frac{g}{l}$ is constant for given pendulum at given place,

$$a \propto y$$

↳ This show simple pendulum executes SHM

Time period (T)

We have, $a = -(g/l)y$

Comparing with $a = -\omega^2 y$

we get, $\cancel{+}\omega^2 = \cancel{+}g/l$

or, $\omega = \sqrt{g/l}$

or, $\frac{2\pi}{T} = \sqrt{g/l}$

$$\boxed{\therefore T = 2\pi\sqrt{l/g}}$$

- Doesn't depend upon mass of bob
- Doesn't depend upon length and acceleration due to gravity

Angular Simple Harmonic Motion

Angular SHM is defined as the oscillatory motion of a body in which the torque (τ) or angular acceleration is directly proportional to the angular displacement (θ) and its direction is opposite to that of angular displacement.

Oscillatory Motion

Oscillatory motion can be termed as the repeated motion in which an object repeats the same movement over and over.

In absence of friction, the oscillatory motion would continue forever which is called free oscillation but in real world, the system eventually settles into equilibrium due to some friction.

Examples of oscillatory motion

1. Free oscillation

When a system or body capable of oscillation is given some initial displacement from its mean position and left free, it begins to oscillate with its own natural frequency with constant amplitude. Then the oscillation is called free oscillation.

eg. Oscillation of pendulum
Vibrating tunning fork or string
Vibration of electric and magnetic fields } In Vaccum

2. Damped oscillation

When due to friction or viscous force or other dissipative force, the amplitudes gradually decreases and come to rest due to resistance in motion of body executing SHM then this type of motion is damped oscillation.

eg. Motion of pendulum in a liquid
To and fro motion of metallic strip in magnetic field

$$\text{Damping} \propto \frac{1}{\text{Peak height of resonance}}$$

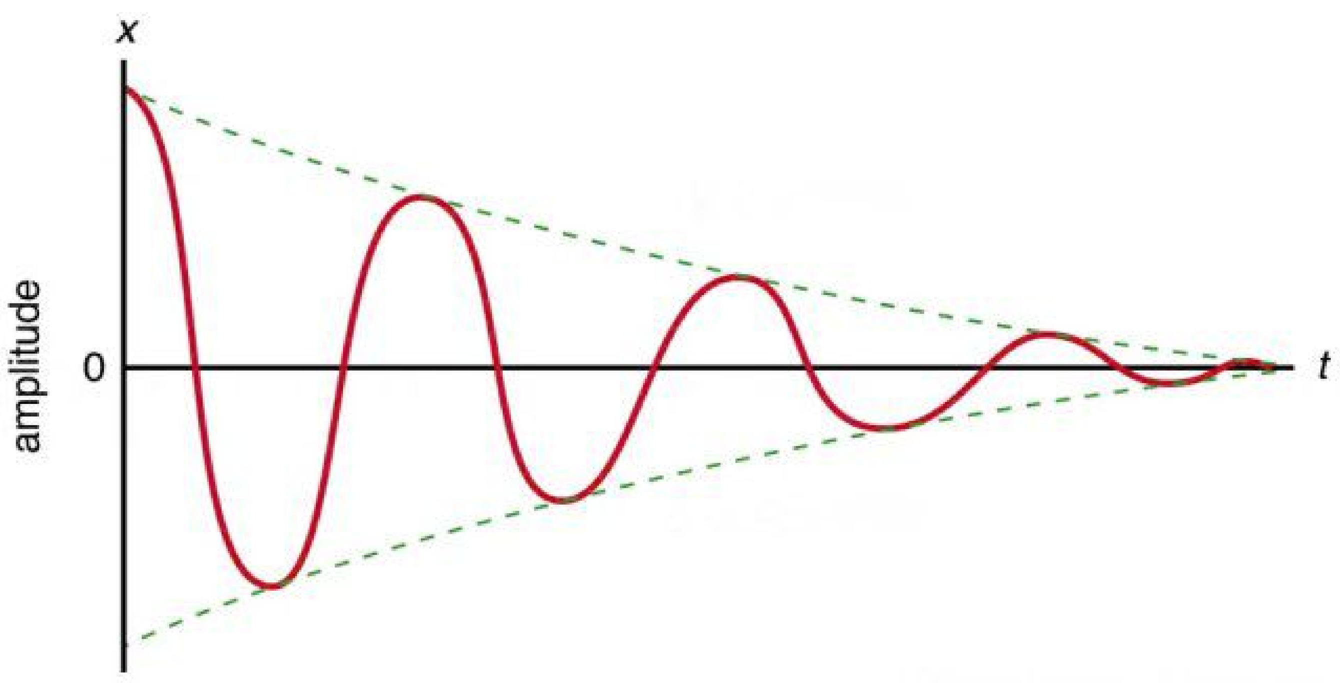

Fig. Damped oscillation

3. Forced vibration and resonant vibration

If a body is set in vibration by an external periodic force whose frequency is equal to the natural frequency of the vibrating body, the amplitude of vibration increases at each step and become very large. Such vibration is called resonant vibrations and phenomena resonance