Fluid Statics

Fluid statics deals with behaviour of fluid when they are at rest and there is no relative motion between fluid particles.

Density (ρ)

The density of substance is defined as mass per unit volume.

$$\rho = \frac{\text{Mass}(m)}{\text{Volume}(V)}$$

Relative density / Specific gravity (ρ_r)

The relative density of substance is the ratio of its density to the density of water at 4°c.

$$\rho_r = \frac{\rho}{\rho_w \text{ at } 4°C}$$

The density of water at 4°c is $1 g/cm^3$.

Thrust

A force acting perpendicular to a surface is called thrust. eg. weight of the liquid in a beaker at the bottom.

Pascal's Law of Pressure

Statement:

When a pressure is applied to an enclosed liquid, the pressure is equally transmitted to every portion of it.

Proof:

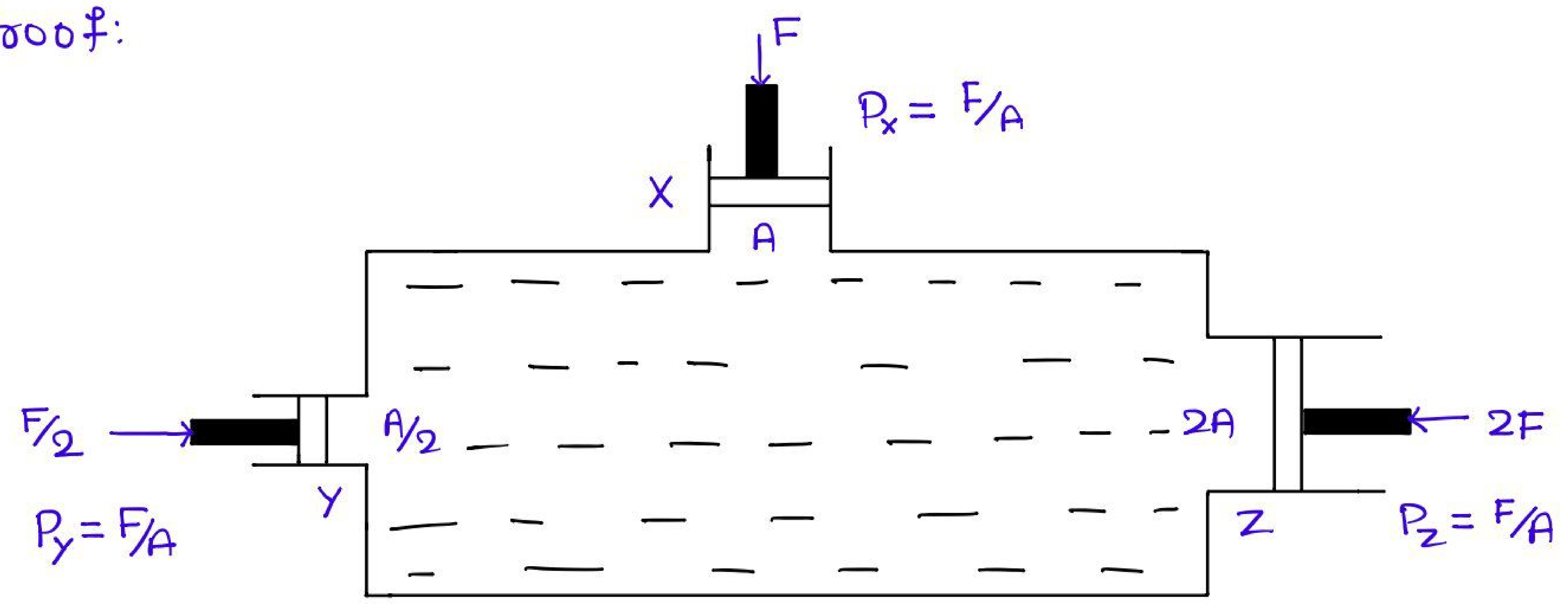


Fig. Pressure on each position in liquid vessel is same

Suppose a vessel containing water with three openings X, Y and Z of different cross section area A, A/2 and 2A respectively. These openings are closed with 3 tight piston to keep water in the vessel. When a force 'F' is applied on X inward, the forces needed to keep the pistons at the same position in Y and Z are F/2 and 2F respectively. Then,

$$\text{Pressure at each opening } (P) = F/A = \frac{F/2}{A/2} = \frac{2F}{2A} = F/A$$

So the pressure is equally transmitted.

Application of Pascal's Law

1. Hydraulic press

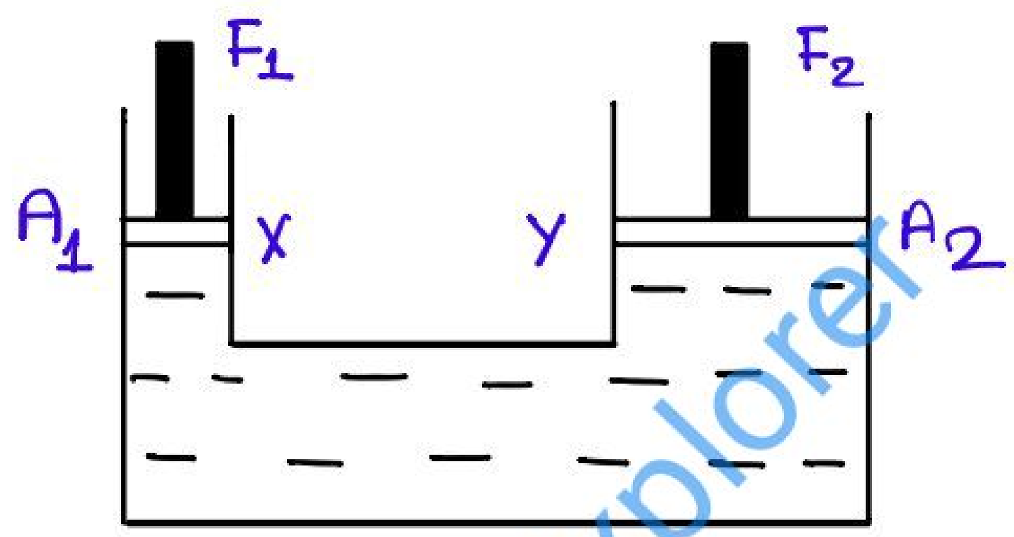


Fig. Hydraulic press

From pascal's law,

$$\text{Pressure at piston } (X) = \text{Pressure at piston } (Y)$$

$$\text{or, } F_1/A_1 = F_2/A_2$$

$$\text{or, } F_2 = \frac{A_2}{A_1} \times F_1$$

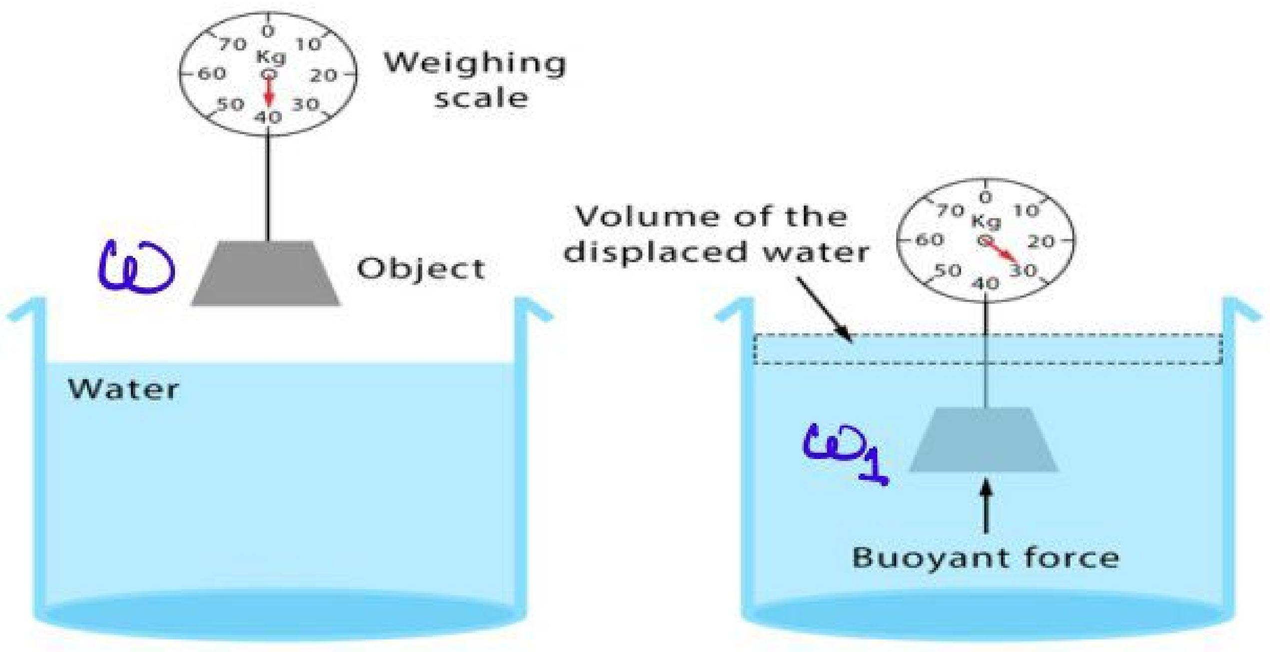
$$\therefore F_2 = \frac{A_2}{A_1} \times F_1$$

2. Hydraulic Jack
3. Hydraulic brake
4. High pressure water jet cutting
5. Teeth scaling

Archimede's Principle

Statement

When the body is fully or partially immersed in a fluid, it experiences an upthrust which is equal to the weight of the fluid displaced by the body.



Upward bouyant force = Weight of the water displaced

suppose a body has weight of 'ω' newton in air and when it is immersed in liquid, the weight of the body in will be ω_1. Then, according to Archimede's principle,

$$\boxed{\text{loss in wt. of body in liquid} = \text{Upthrust} = \omega - \omega_1}$$

Upthrust or Buoyancy

The upward force exerted by a fluid on an object which is completely or partially immersed in the fluid is called upthrust or buoyancy. Object has lesser weight in fluid.

Pressure in fluid

The force per unit area acting normally on a surface.

$P = F/A$

For liquid,

$$P = F/A = \frac{\text{wt. of liquid}}{A} = \frac{mg}{A} = \frac{\rho \times V \times g}{A} = \frac{\rho \times \cancel{A} \times h \times g}{\cancel{A}} = \rho h g$$

$$\boxed{\therefore P = h\rho g}$$

Principle of floatation

The fluid in which body floats should relocate or displace the fluid of its own weight to float.

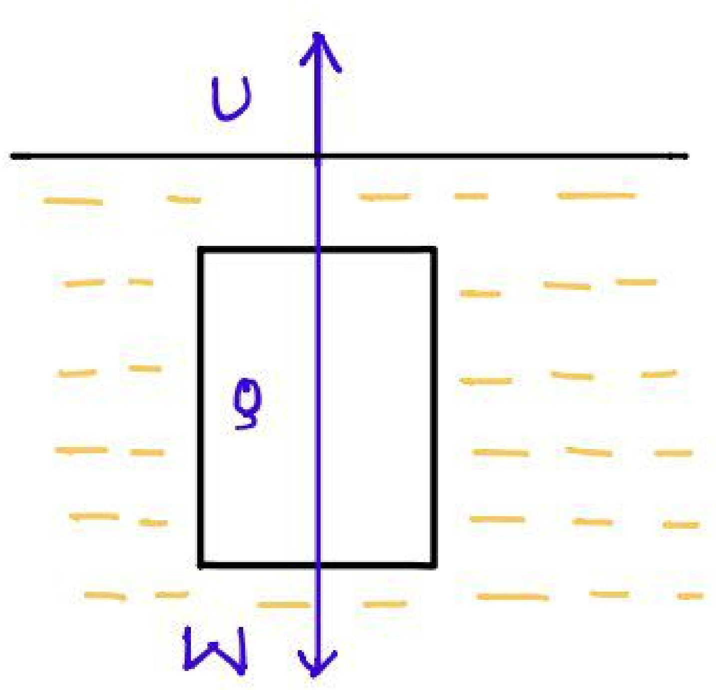


Fig.1 A body complete inside liquid
$\rho > \rho_l$

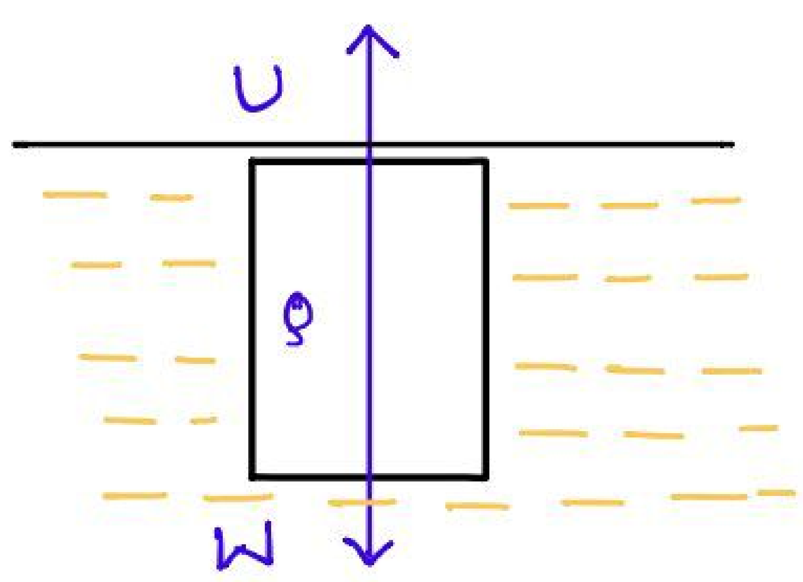


Fig.2 A body inside liquid
$\rho = \rho_l$

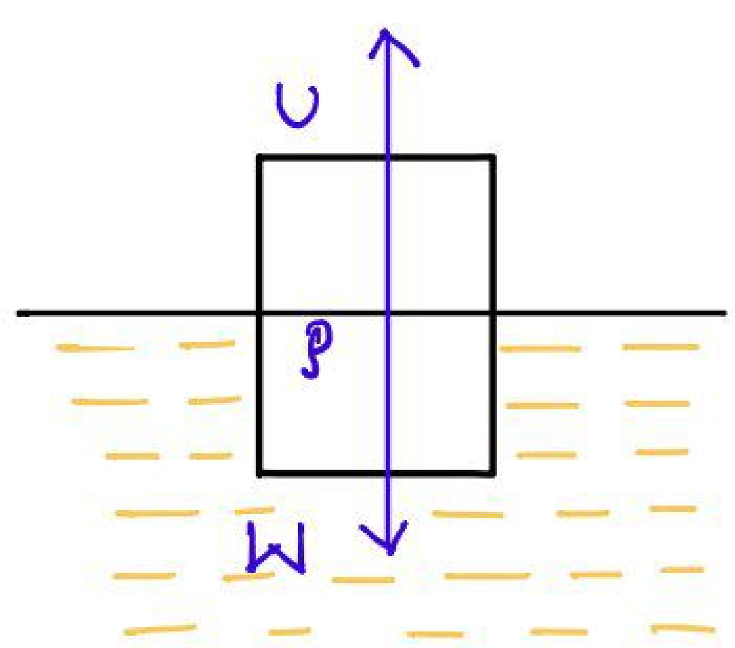


Fig.3 A body floating in liquid
$\rho < \rho_l$

Case 1

If a body is completely immersed in liquid, its weight 'w' acts vertically downward and the upthrust 'U' of displaced liquid acts vertically upward. If the weight of body of volume 'V' is greater than the upthrust, the body will sink in liquid and will lie at the bottom of the container.

So, $w > U$

or, $\rho \cancel{V} g > \rho_l \cancel{V} g$

$\boxed{\therefore \rho > \rho_l}$

where,

ρ = density of body

ρ_l = density of liquid

Hence, the body will sink in a liquid if the density of body is greater than the density of liquid as in fig. 1

Case 2

If $w = U$, the body just sinks and remain inside the liquid with upper surface near the liquid as in fig. 2

So, $w = U$

or, $\rho \cancel{V} g = \rho_l \cancel{V} g$

$\boxed{\therefore \rho = \rho_l}$

Case 3

If the weight of the body is smaller than the upthrust i.e $w < U$, the body will float on the surface of liquid as in fig. 3

So, $w < U$

or, $\rho \cancel{V} g < \rho_l \cancel{V} g$

$\boxed{\therefore \rho < \rho_l}$

Note:

A body will float in liquid if its density is smaller than liquid. So a floating body displaces the liquid of its own weight. Then,

$U = mg$

or, $\rho_l V_l g = \rho V g$

or, $V_l / V = \rho / \rho_l$

where,

V_l = Volume of displaced liquid which is equal to volume of body inside liquid

Surface Tension

The property of liquid at rest by the virtue of which it's surface behaves like a stretched membrane and tries to occupy minimum possible surface area is called surface tension.

or

It is the force per unit length of an imaginary line drawn in plane of liquid surface acting right angle to this line.

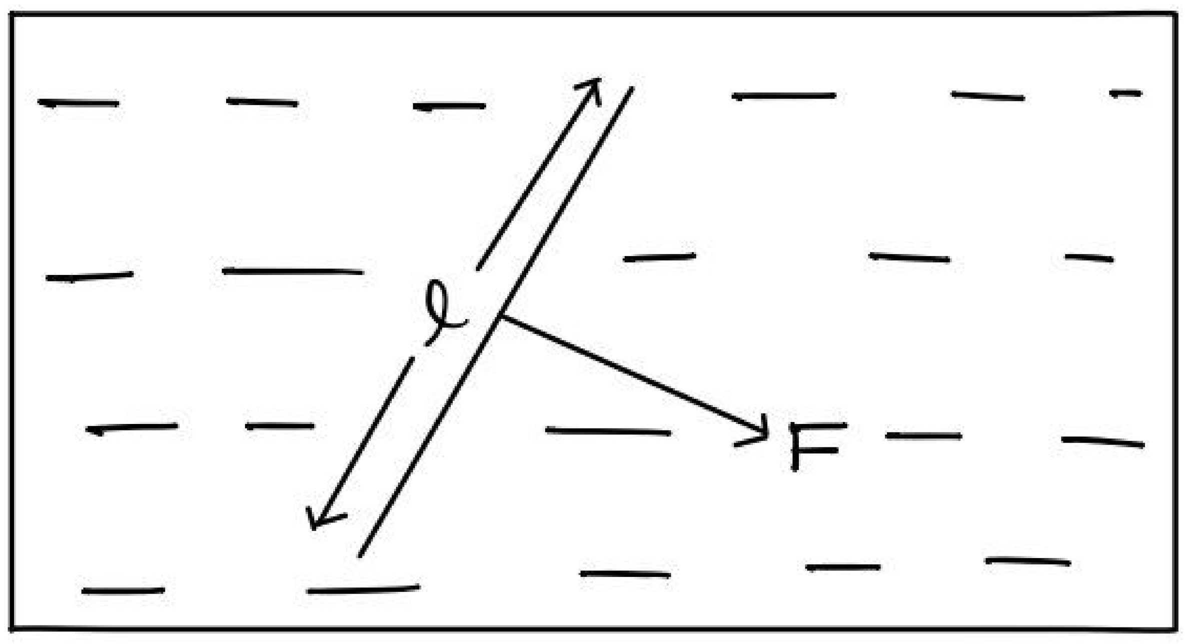


Fig. surface tension on free surface of liquid

If F be the imaginary line of length 'l' Then,

$$\boxed{\text{surface tension } (T) = F/l} \text{ ——— eqn. ①}$$

Unit - N/m (SI)

Dyne/cm (CGS)

Dimension - $\frac{[MLT^{-2}]}{[L]} = [MT^{-2}]$

From eqn ①

$$F = T \times l$$

If a thread of length 'l' floats on liquid, the total length in contact with liquid surface is 2l. Then,

Force of surface tension $(F) = T \times 2l$

If a ring of negligible thickness and radius 'r' floats on liquid surface, then length of ring in contact with liquid surface is $(2 \times 2\pi r)$

so,

$$F = T \times 2(2\pi r)$$
$$= 4\pi r T$$

Molecular Theory of surface Tension

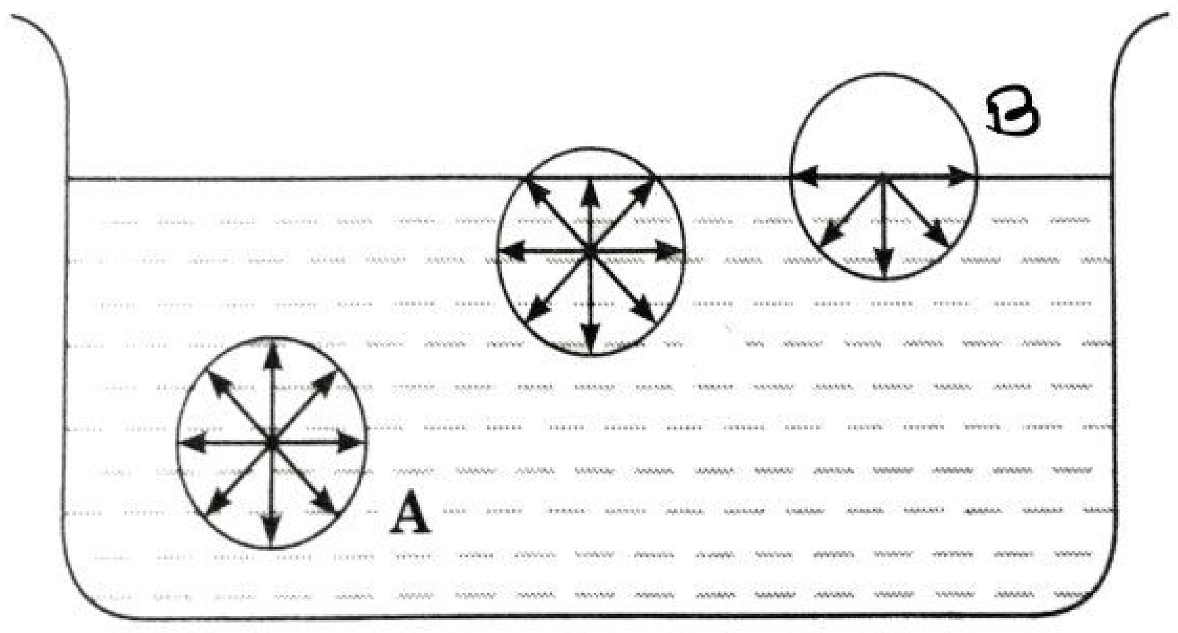


Fig. Molecular force in liquid

Consider a molecule 'A' of liquid lying well below the free surface of liquid. As this molecule is attracted by the neighbouring molecules lying within the sphere, the resultant force due to all the molecules on A is zero. Consider another molecule B on the surface of the liquid, the molecule B experiences force of attraction due to the molecules lying in the lower half of the sphere. The resultant of all of these forces is downward. So the molecules on the surface experiences maximum downward force. For equilibrium, a system must have minimum potential energy, so there must be minimum number of molecules on the liquid surface which is possible only when the liquid surface is minimum. That is why the liquid surface contracts like stretched elastic membrane.

Some examples explaining surface tension

- Thread on a soap film
- Floating needle
- When a dry brush is dipped into water, its hair spread out

Surface Energy

We know that free surface of liquid always has tendency to contract and occupies minimum surface area. If the surface area of liquid has to be increased, work has to be done. The work done is stored in liquid surface film as its potential energy.

The potential energy per unit area of the surface film is called surface energy.

It is also defined as that the amount of work done in increasing the area of surface film through unity.

$$\text{Surface energy}(\sigma) = \frac{\text{work done in increasing surface area}(W)}{\text{Increase in surface area}(\Delta A)}$$

Relation between surface tension and surface energy

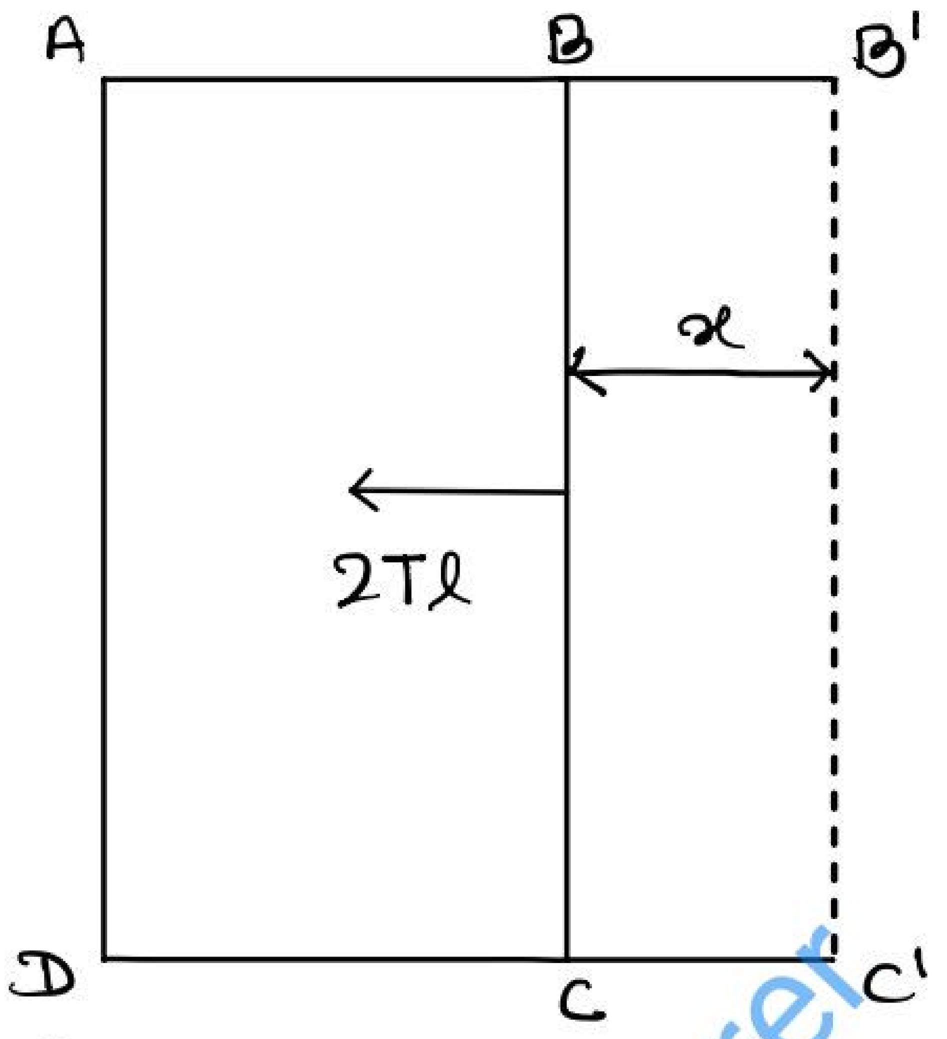


Fig. surface energy and surface tension

Consider a rectangular frame ABCD of wire in which BC is moveable. Now, dip the frame in soap solution, a thin film is formed which pulls the wire BC towards left due to surface tension. If 'T' is surface tension of the film and 'l' is the length of wire BC, then the force 'F' due to surface tension is given by,

$$F = T \times \underline{2l}$$

↳ As the film has two surfaces in contact with air so total length of wire (BC) is $2l$.

Suppose the wire is now moved through a distance 'x' from BC to B'C' against surface tension force 'F' so that the surface area of film increase. In order to increase the film area, work has to be done against F.

$$\begin{aligned}\text{work done in increasing surface area} &= F \times \text{distance}\\ &= 2Tlx\end{aligned}$$

$$\text{Increase in surface area} = 2(l \times x) = 2lx$$

Now,

$$\text{surface energy}(\sigma) = \frac{W}{\Delta A} = \frac{\cancel{2}T\cancel{l x}}{\cancel{2lx}} = T$$

$$\therefore \boxed{\sigma = T}$$

↳ Thus, surface tension is numerically equal to surface energy

Excess pressure on curved surface of a liquid

For curved surface, the pressure on its convex side is less than pressure in concave side.

Excess pressure inside a liquid drop

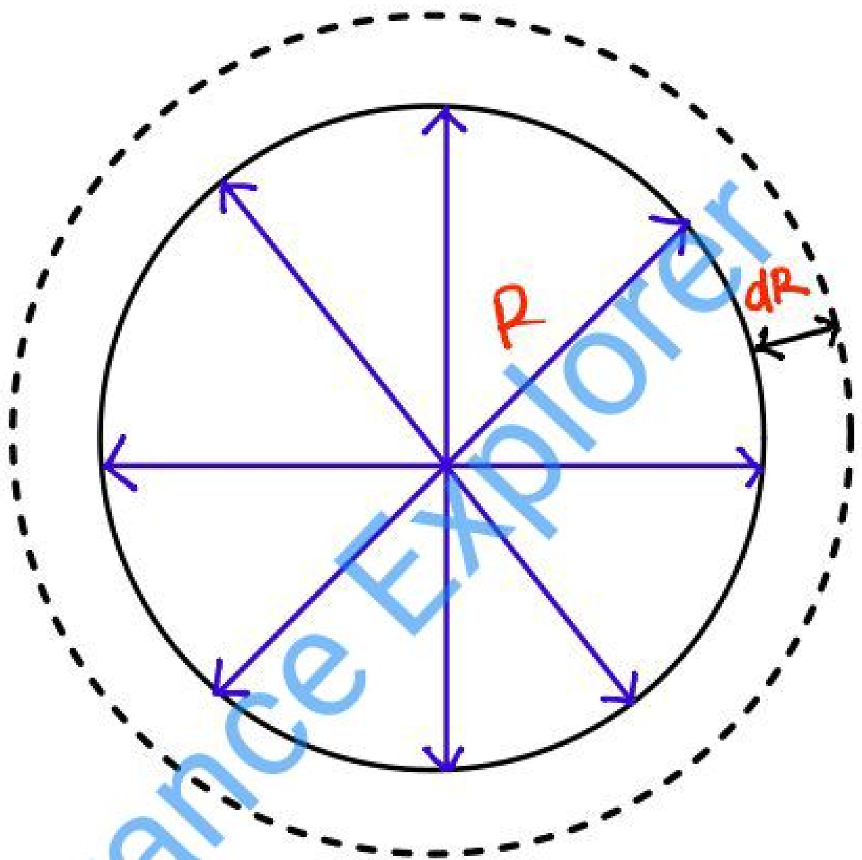


Fig. Excess pressure inside a liquid drop

Consider a drop of radius 'R' the molecules lying on the surface of liquid drop due to surface tension will experience a resultant force acting inwards perpendicular to the surface. As a result the pressure inside the drop must be greater than pressure pressure outside it. The excess pressure inside the drop will provide a force acting outwards perpendicular to the surface to balance the resultant force due to surface tension.

Let 'T' be the surface tension and 'P' be the excess pressure inside the drop. Suppose due to excess pressure, there be increase in radius by dR. So, work done by excess pressure,

$$W = \text{Force} \times \text{displacement}$$

$$= (\text{Excess pressure} \times \text{Area}) \times \text{Increase in radius}$$

$$\therefore W = P \times 4\pi R^2 \times dR \quad \text{———— eq}^n. (1)$$

Increase in surface area = Final surface area − Initial surface area

$$= 4\pi(R+dR)^2 - 4\pi R^2$$
$$= 4\pi(R^2 + 2RdR + dR^2) - 4\pi R^2$$
$$= \cancel{4\pi R^2} + 8\pi RdR + 4\pi dR^2 - \cancel{4\pi R^2}$$

since dR is very small it is neglected

$\therefore \Delta A = 8\pi RdR$

$\therefore$ Increase in surface energy = $\Delta A \times$ surface tension

$$= 8\pi RdR \times T$$

Since increase in surface energy is the work done by the excess pressure, so

work done $= 8\pi RdRT$ —— eqn.②

From eqn.① and eqn.②

$$P \times \cancel{4\pi} R^{\cancel{2}} \times \cancel{dR} = \overset{2}{\cancel{8}}\cancel{\pi R}\,\cancel{dR}\,T$$

or, $P = \frac{2T}{R}$

$$\boxed{\therefore P_{in} - P_{out} = \frac{2T}{R}}$$

Excess pressure of air bubble

Since, air bubble has only one free surface, so excess pressure is given by

$$\boxed{\therefore P_{in} - P_{out} = \frac{2T}{R}}$$

Excess pressure inside liquid bubble or soap bubble

Since, liquid bubble has two free surface, so excess pressure is given by

$$\boxed{P_{in} - P_{out} = 2 \times \frac{2T}{R} = \frac{4T}{R}}$$

Shape of liquid surface

The surface of liquid is usually curved where it is in contact with a solid. The curved surface of liquid is called meniscus. The shape of meniscus is determined by relating strength of cohesive and adhesive force acting on the molecule.

At convex meniscus,

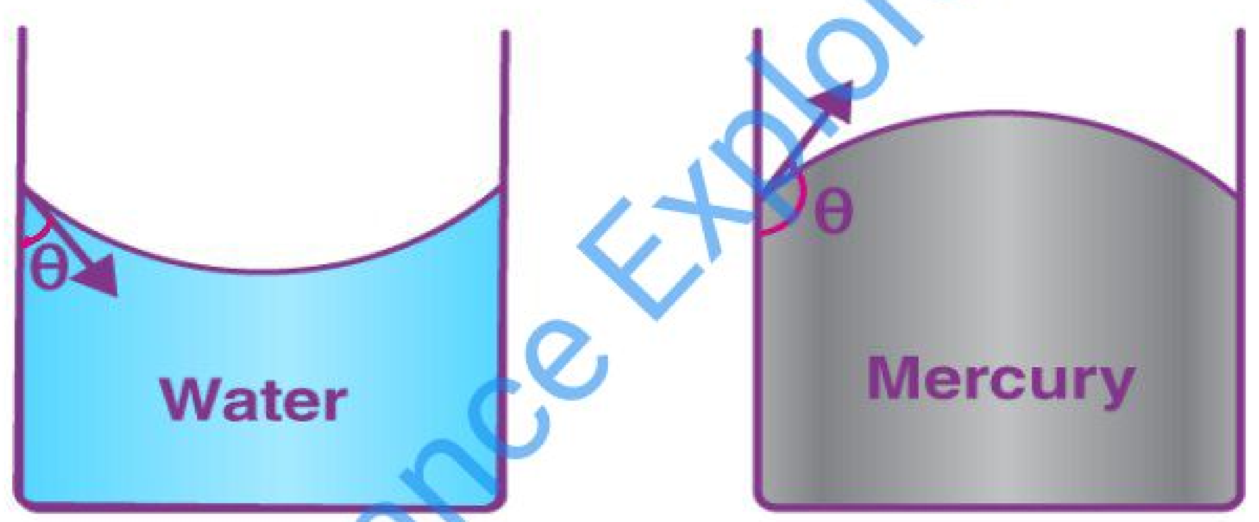
Cohesive force > Adhesive force

At concave meniscus,

Cohesive force < Adhesive force

Angle of contact

When a liquid is in contact with a solid, the angle between the tangent drawn to the free surface of the liquid and the surface of the solid at the point of contact measured inside the liquid is called angle of contact. It is denoted by θ.



(A) Acute angle (B) Obtuse angle

wet wall of container ↲ ↳ Do not wet wall of container

• The addition of detergent to liquid lowers its surface tension and decrease angle of contact

Angle of contact for,

Pure water and clean glass – $0°$

Ordinary water and glass – $8°$ (acute)

Mercury and glass – $140°$ (obtuse)

Angle of contact depends on

1) Nature of liquid and solid in contact

2) The medium that exists above free surface of liquid

Capillarity

The rise or fall of liquid in a tube of very fine bore is called capillarity or capillary action.

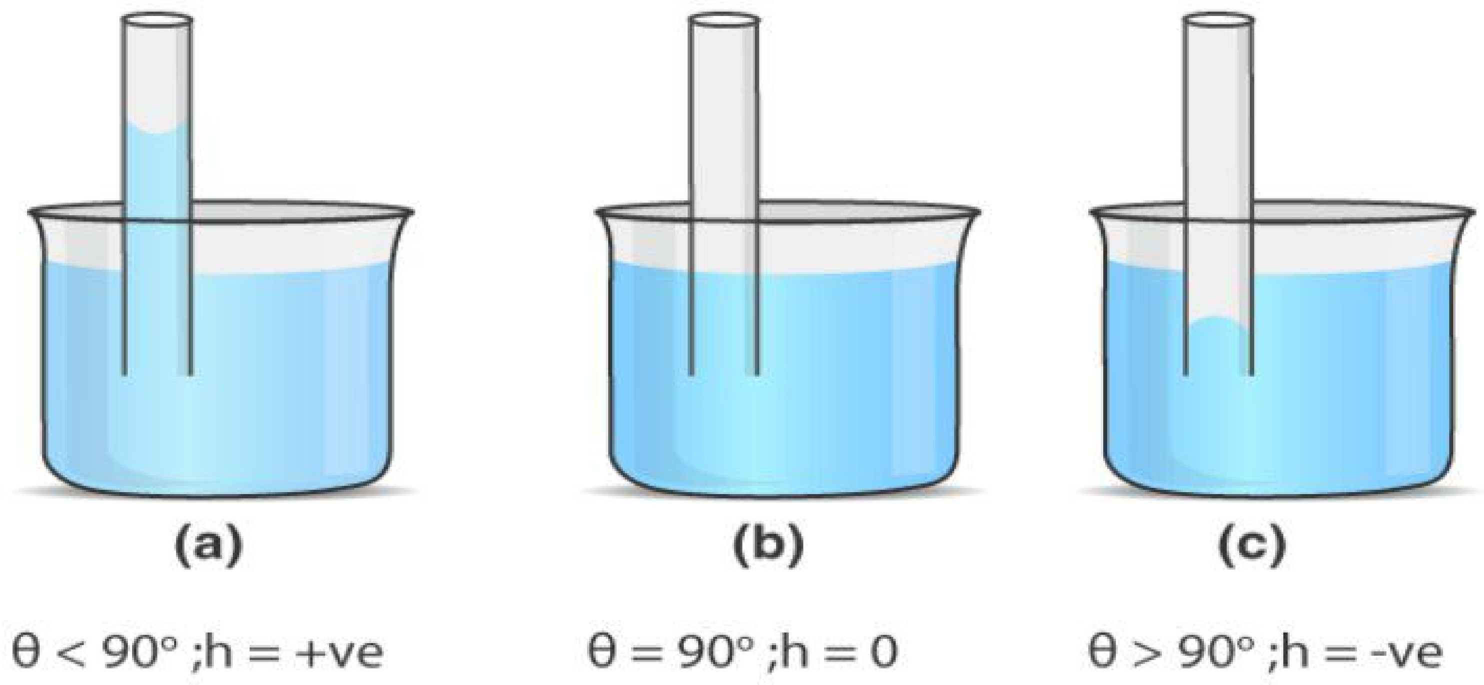


Fig. 1 Fig. 2 Fig. 3

- A liquid whose angle of contact is less than 90°, suffers capillary rise as in fig 1 which consist of water and angle of contact is less than 90°.
- A liquid whose angle of contact is more than 90° suffers capillary depression as in fig. 3 which consist of mercury and angle of contact is more than 90°.
- If angle of contact is 0°, the liquid will neither rise nor fall as in fig. 2.

Examples of capillarity

- Oil rises in cotton wicks of lamps through the small capillaries between threads.
- A blotting paper absorbs ink by capillary action.

Measurement of Surface Tension by Capillary Rise Method

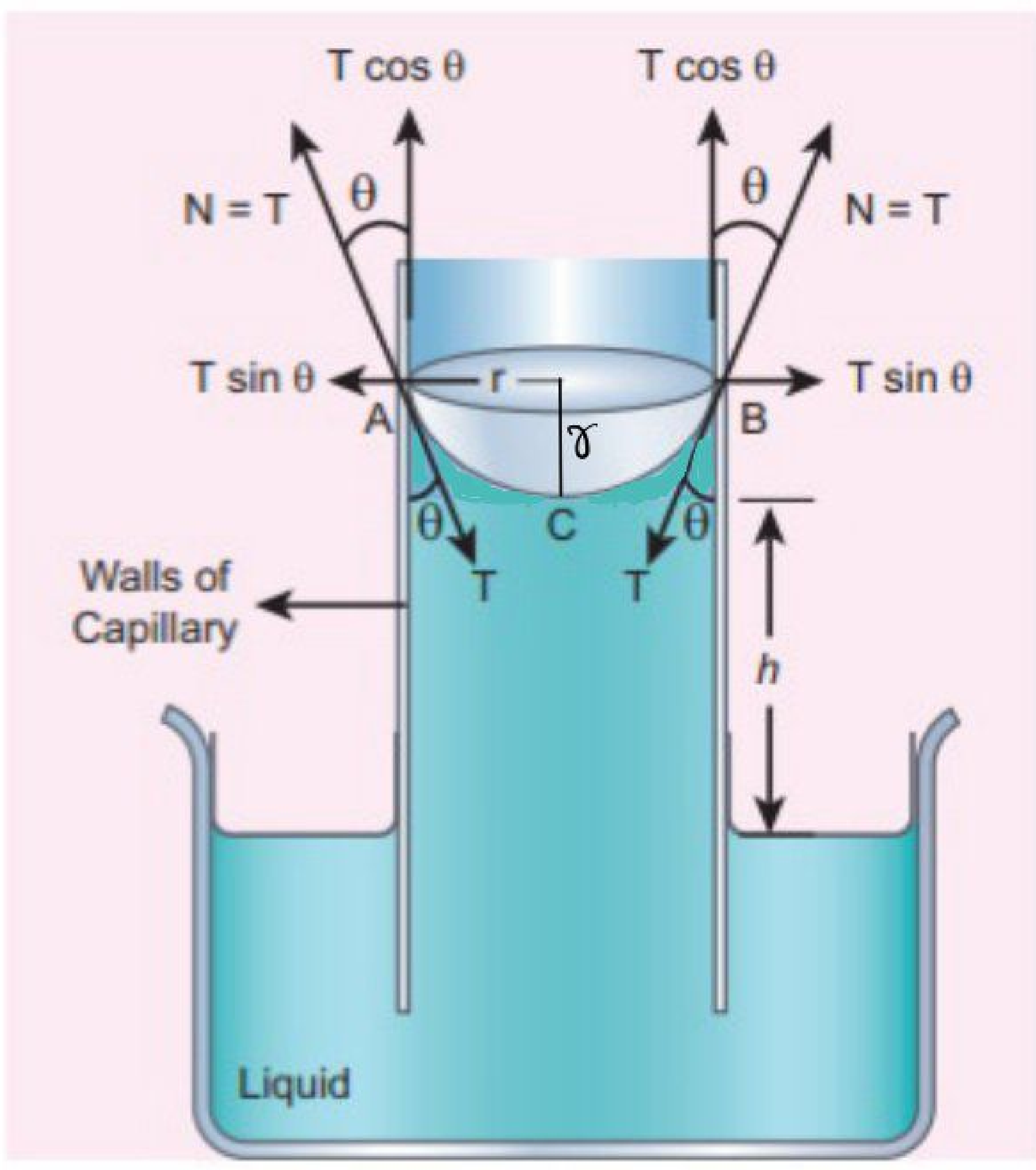


Fig. Capillary rise method

Consider a capillary tube of radius 'r' open at both ends and dipped into liquid (eg. water) which has concave meniscus. Let 'θ' be the angle of contact, 'h' be the height of liquid rises, 'ρ' be the density of liquid and 'T' be the surface tension of liquid.

The surface tension forces causes the liquid to exert a downward directed force 'T' on the walls of the tube. This force 'T' acts along the tangent at the point of contact A, B. From Newton's third law of motion, the tube exerts an equal and opposite reaction 'N'. The reaction (N=T) can be resolved into two components Tsinθ (along horizontal) and Tcosθ (along vertical). The horizontal components cancel to each other whereas the vertical components are added which pulls the liquid upward. The component Tcosθ acts along the whole circumference of meniscus.

$$\text{Total upward force} = T\cos\theta \times 2\pi r$$

Volume of liquid in tube above free surface of liquid (V) = Volume of cylinder of height 'h' and radius 'r' + Volume of cylinder of height 'r' and radius 'r' — Volume of hemispher of radius 'r'

$$V = \pi r^2 h + \pi r^2 r - \frac{1}{2}\cdot\frac{4}{3}\pi r^3$$

$$= \pi r^2 h + \pi r^3 - \frac{2}{3}\pi r^3$$

$$= \pi r^2 h + \frac{\pi r^3}{3}$$

$$\therefore V = \pi r^2 (h + r/3)$$

Weight of liquid column = Mass of liquid column × g

$$= \rho \times V \times g$$

$$= \rho \pi r^2 (h + r/3) g$$

For equilibrium,

Total upward force = Weight of liquid column

$$\text{or, } T\cos\theta \times 2\pi r = \rho \pi r^2 (h + r/3) g$$

$$\text{or, } T = \frac{\rho r (h + r/3) g}{2\cos\theta}$$

When the tube is of very fine bore, r/3 may be neglected compared to height 'h' so surface tension becomes,

$$\text{or, } T = \frac{\rho r h g}{2\cos\theta}$$

$$\boxed{\therefore h = \frac{2T\cos\theta}{r\rho g}}$$

↳ This expression is called ascent formula. In this case angle of contact is acute. Note that narrow the tube, the greater is the height to which the liquid rises.

Viscosity

The property of fluid by virtue of which an internal friction comes into play when fluid is in motion and opposes the relative motion of its different layers is called viscosity.

Newton's law of Viscosity

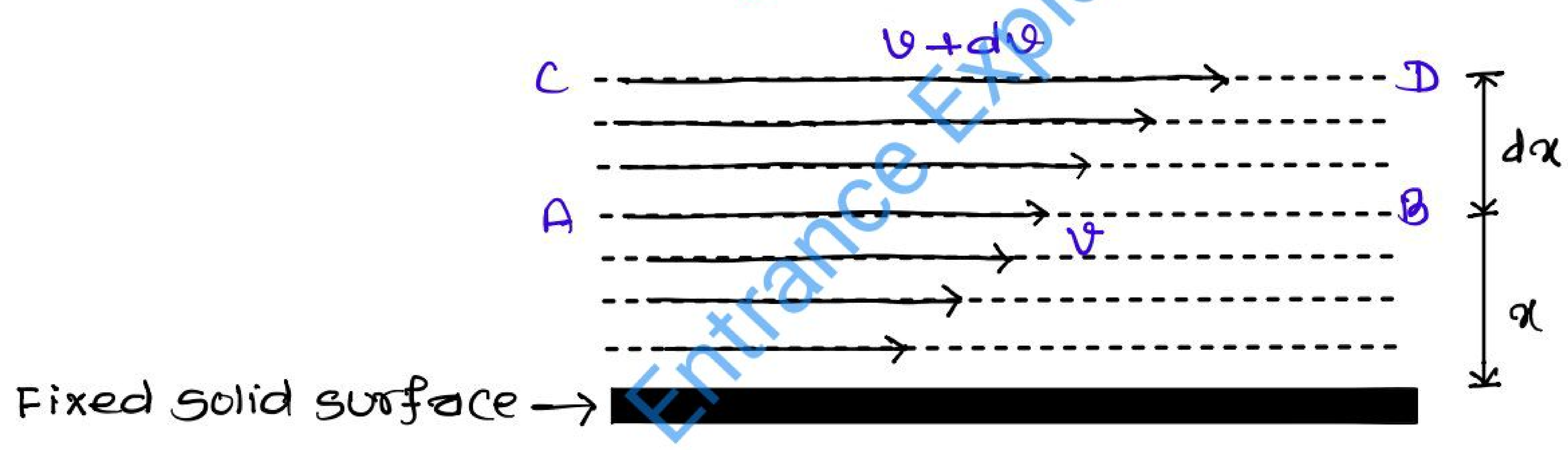


Fig. Different layers of liquid flowing over solid surface

Consider a liquid flowing steadily over a fixed solid horizontal surface. The layer of liquid moves parallel to the fixed surface. The layer in contact with the fixed surface is constant while the velocities of other layers increases uniformly upwards. Consider two layers AB and CD moving with velocities v and dv at a distance 'x' and '$x + dx$' from fixed surface respectively.

According to Newton, the viscous force 'F' depends upon

i) Directly proportional to area 'A' of the layers in contact i.e

$$F \propto A$$

ii) Directly proportional to velocity gradient between the layers i.e

$$F \propto \frac{dv}{dx}$$

Combining these two factors,

$$F \propto A \frac{dv}{dx}$$

or, $\boxed{F = -\eta A \frac{dv}{dx}}$ ———— eqn ①

where,

η = coefficient of viscosity (It's value depend upon nature of liquid)

(-ve) sign shows that the direction of viscous force (F) is opposite to show direction of motion of liquid.

If $A = 1$ and $\frac{dv}{dx} = 1$, Then from eqn ① we have, $\eta = -F$

So, coefficient of viscosity of liquid is defined as the viscous force acting per unit area of the layer having unit velocity gradient perpendicular to the direction of flow of the liquid.

Unit and dimension of η,

$$\eta = \frac{F}{A\left(\frac{dv}{dx}\right)} = \frac{\text{Newton}}{m^2 \frac{m/s}{m}} = \text{Newton } m^{-2} sec$$

$$= \text{Dyne } cm^{-2} sec$$

$$= \text{Poise}$$

$$1 \text{ poise} = 1 \text{ dyne } cm^{-2} sec = 10^{-5} N \times 10^{4} m^{-2} \times 1 sec$$

$$= 0.1 \, N m^{-2} sec$$

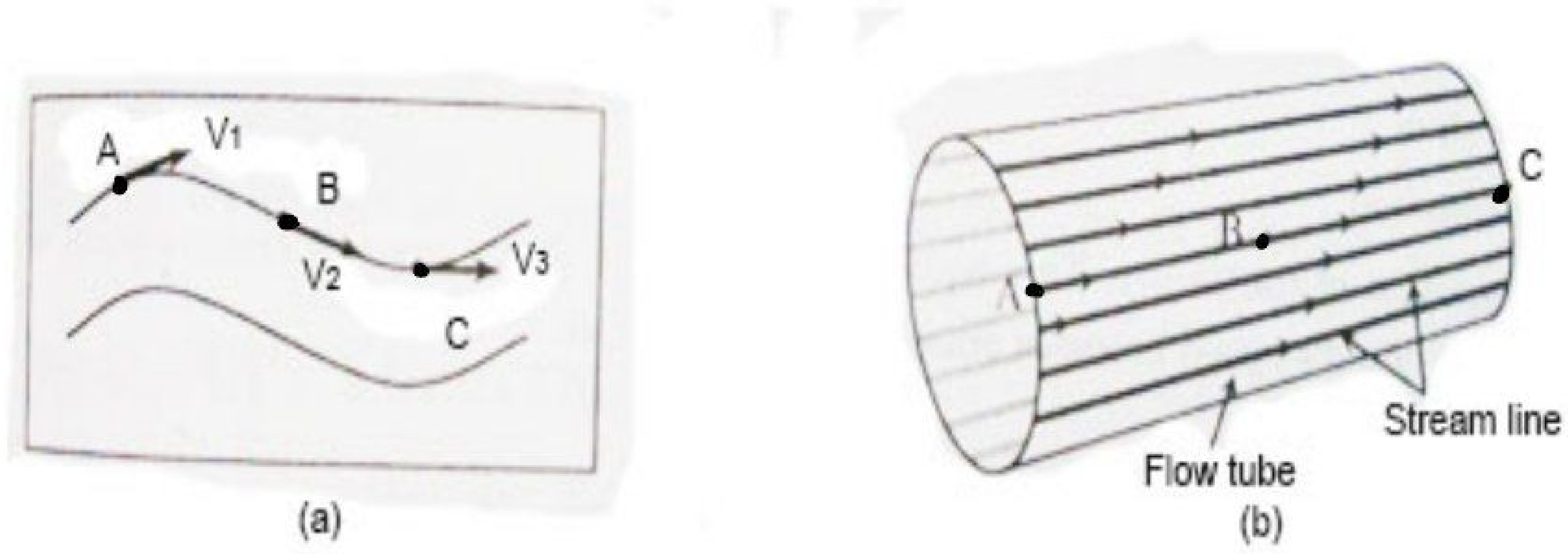
$$\boxed{1 \text{ decapoise} = 10 \text{ poise} = 1 \, N m^{-2} sec}$$

$$\text{Dimension} = \frac{[MLT^{-2}]}{[L^2]\left[\frac{LT^{-1}}{L}\right]} = [ML^{-1}T^{-1}]$$

SI unit = $kg \, m^{-1} s^{-1}$

Streamline and Turbulent Flow

When the flow of liquid is such that the velocity of every particle at any point of the fluid is constant, then the flow is said to be steady or streamline flow.



(a) Stream-line in a liquid (b) Stream line flow of liquid.

Consider a liquid flowing through a tube. let the path followed by the particles of the liquid is represented by a line ABC. Let v_1, v_2 and v_3 be the velocities of a particle of the liquid at point A, B and C respectively on its path. If all the succeeding particles of the fluid move along ABC with velocities v_1, v_2 and v_3 at A, B and C respectively, the flow of liquid is steady or stream-line flow. The path is known as steady or streamline. Two streamline never cross to each other.

Laminar flow

If a liquid is flowing over a horizontal surface with steady flow and flow in the form of layer of different velocities which do not mix with each other, then the flow of liquid is called laminar flow.

Turbulent flow

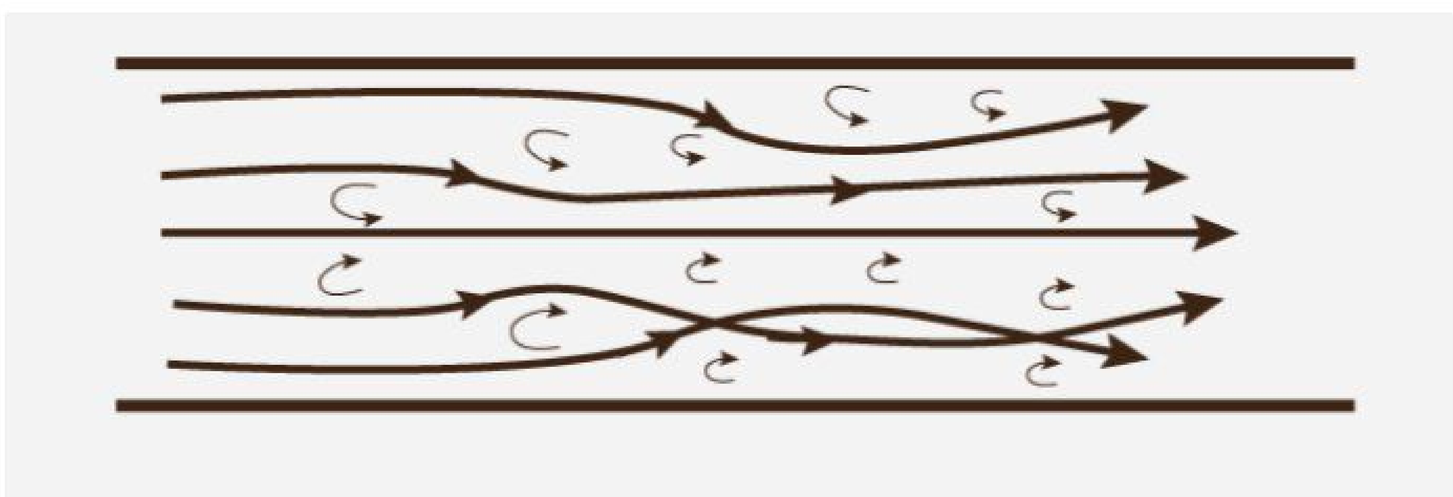


Fig. Turbulent flow

When a liquid moves with velocity greater than its critical velocity, the motion of the particle of liquid becomes disorderly or irregular. Such a flow is called turbulent flow.

eg. Smoke from cigarette rising a short distance
Flow of water just behind boat or ship
Air flow behind moving train

Critical Velocity and Reynold's Number

When the fluid flows through a tube with small velocities, motion will be streamline but on increasing the velocity when it becomes greater than certain limiting value, the motion becomes turbulent. The limiting velocity below which the fluid is streamline and above which it is turbulent is critical velocity.

The critical velocity (v_c) depends on

i) Coefficient of viscosity of fluid (η)

ii) Density of the fluid (ρ)

iii) Lateral dimension of the tube (r)

Now using dimension,

Let, $v_c = K \eta^a \rho^b r^c$

where, K = dimensionless constant

$$[LT^{-1}] = [ML^{-1}T^{-1}]^a [ML^{-3}]^b [L]^c$$

or, $LT^{-1} = M^{a+b} L^{-a-3b+c} T^{-a}$

According to principle of homogeneity of dimension,

$a+b=0$ $\quad\therefore a = 1$

$-a-3b+c=1$ $\quad b=-1$

and $-a=-1$ $\quad c=-1$

Hence,

$$v_c = K\eta^1 \rho^{-1} r^{-1} = K\frac{\eta}{\rho r}$$

$$\boxed{\therefore\ v_c = K\frac{\eta}{\rho r}}$$

↳ It is called Reynold's formula

where, K = Reynold's number

- For streamline flow, critical velocity should be large and for large critical velocity, η must be large, ρ and r must be small.

- For narrow tubes, Reynold's number is 1000. After exceeding this number the flow becomes turbulent. Thus, it is pure numerical value and is independent of the system of units used.

Poiseville's formula

- Studied the streamline flow of fluid

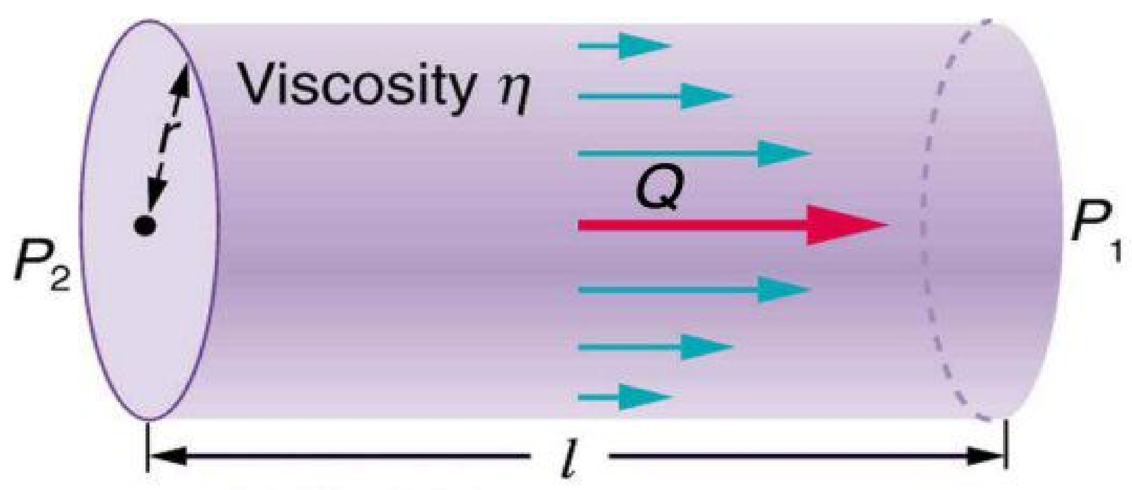


Fig. Streamline flow of liquid

- The volume of liquid flowing per second (Q) through a capillary tube is

1) Directly proportional to the difference of pressure (ΔP) between the two ends of the tube

$$Q \propto \Delta P$$

2) Directly proportional to the fourth power of radius (r) of capillary tube

$$Q \propto r^4$$

3) Inversely proportional to the length (l) of capillary tube

$$Q \propto 1/l$$

Combining these factors, we get

$$Q \propto \frac{\Delta P r^4}{\eta l}$$

or, $$Q = \frac{\pi \Delta P r^4}{8 \eta l}$$

$$\therefore \boxed{Q = \frac{\pi \Delta P r^4}{8 \eta l}}$$ $\longrightarrow$ Required Poiseville's equation

Where,

$K = \frac{\pi}{8}$, a proportionality constant

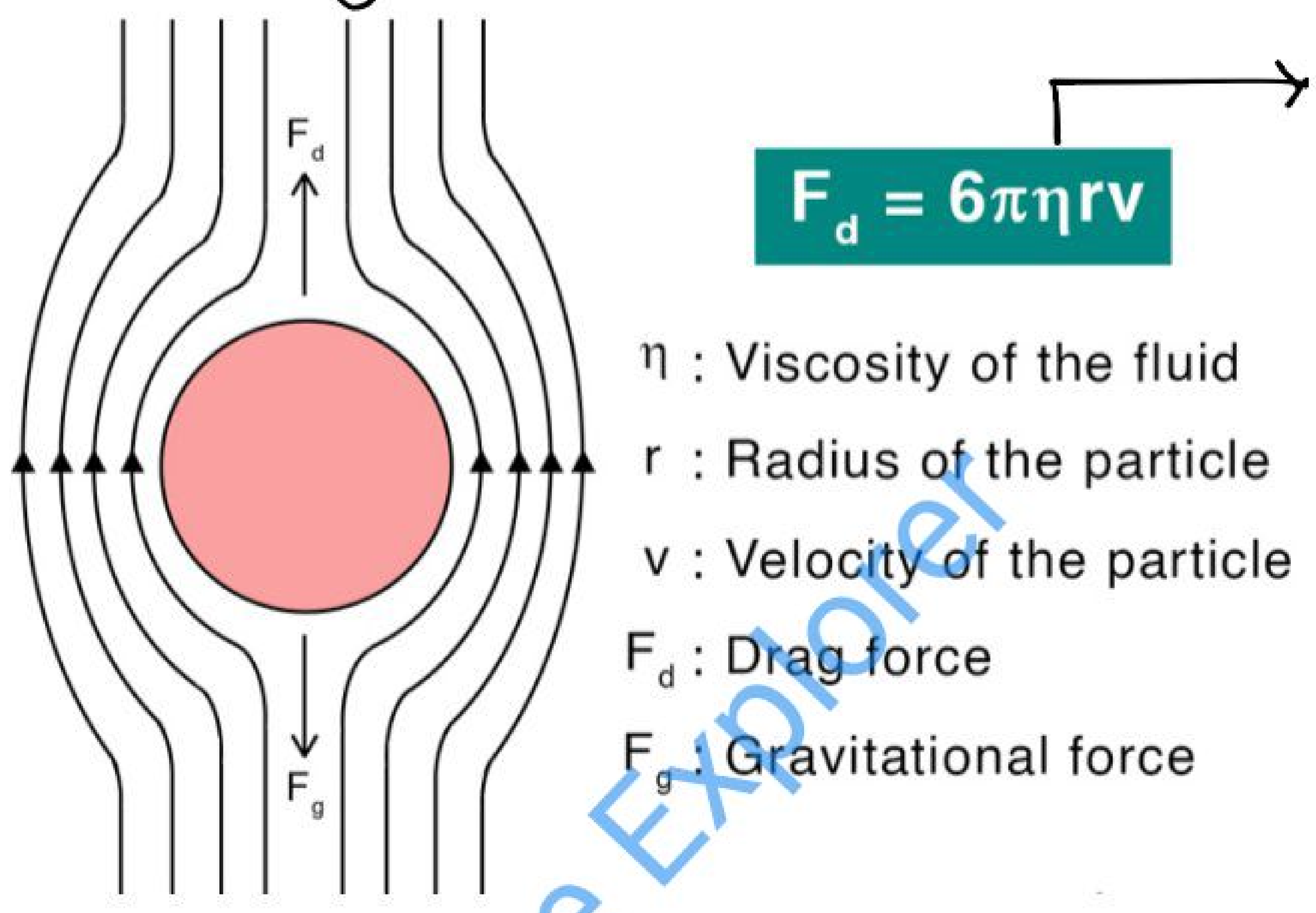
Applications:

1. Used to study the fluid feeding by insects that have sucking mouth parts.
2. Determine the blood flow through the veins in the body.
3. Describe the mineral melt motion in fibre production.
4. It is applied to the flow of liquid through drinking straw.

Stoke's law

When a small spherical body falls through a viscous fluid, the layer of fluid in contact with the body also moves with the same velocity. But the layers of fluid at a large distance from falling body remain undisturbed. Thus, the falling body produces a relative motion between different layers of fluid. As the velocity increases the opposing force also increases.

Body after attaining a certain velocity starts moving with constant velocity in the fluid. This constant velocity is called terminal velocity.



Required stokes formula

Fig. A body falling in fluid produces a relative motion between different layers of fluid

Applications:

1. Determination of coefficient of viscosity of liquid.
2. Determination of value of charge on an electron.
3. While jumping from an airplane, parachute help us to land safely on the earth.
4. Rain drops do not acquire high velocity during free fall, if this does not happen a person moving in rain would get hurt.

Determination of viscosity of liquid using stoke's law

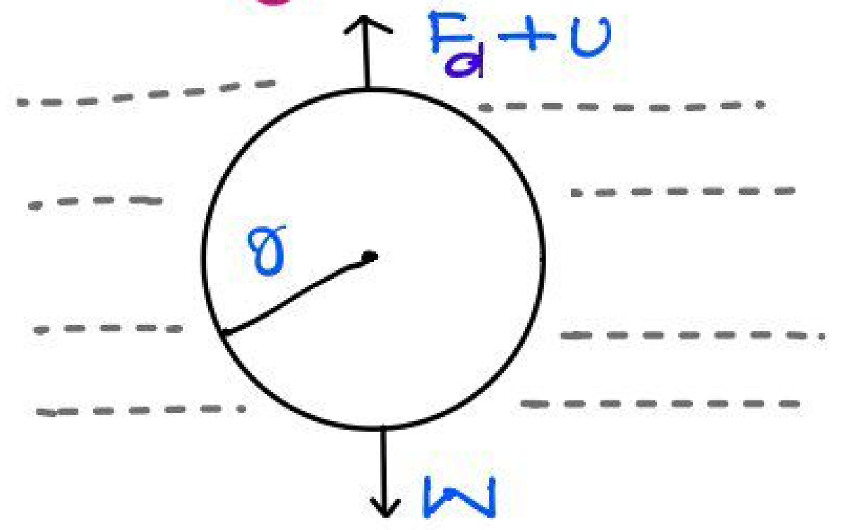


Fig. A ball falling through viscous liquid due to gravity

Let 'r' be the radius of ball falling through the viscous fluid of density 'σ' and coefficient of viscosity 'η'. Let 'ρ' be the density of material of the ball. The various forces acting on the body are:

a. It's weight (W) in downward direction
b. Upward thrust (U) equal to weight of displaced fluid
c. Viscous drag force (F_d) in direction opposite to direction of motion of body

When ball is falling with terminal velocity (v) Then,

Downward force = upward force

or, $W = F_d + U$

or, $mg = 6\pi r \eta v$ + weight of liquid displaced

where, $F_d = 6\pi r \eta v$ → from stoke's law

or, $\rho \times V \times g = 6\pi r \eta v + \sigma \times V \times g$

or, $6\pi r \eta v = \rho \times V \times g - \sigma \times V \times g$

or, $6\pi r \eta v = Vg(\rho - \sigma)$

or, $\cancel{6}^{3}\cancel{\pi}\cancel{r}\eta v = \frac{\cancel{4}^{2}}{3}\cancel{\pi} r^{\cancel{3}} g(\rho - \sigma)$

or, $\eta = \frac{2}{9}\frac{r^2}{v} g(\rho - \sigma)$

$$\boxed{\therefore \eta = \frac{2}{9}\frac{r^2}{v}(\rho - \sigma) g}$$

↳ This is the expression for coefficient of viscosity of liquid or fluid

Equation of Continuity

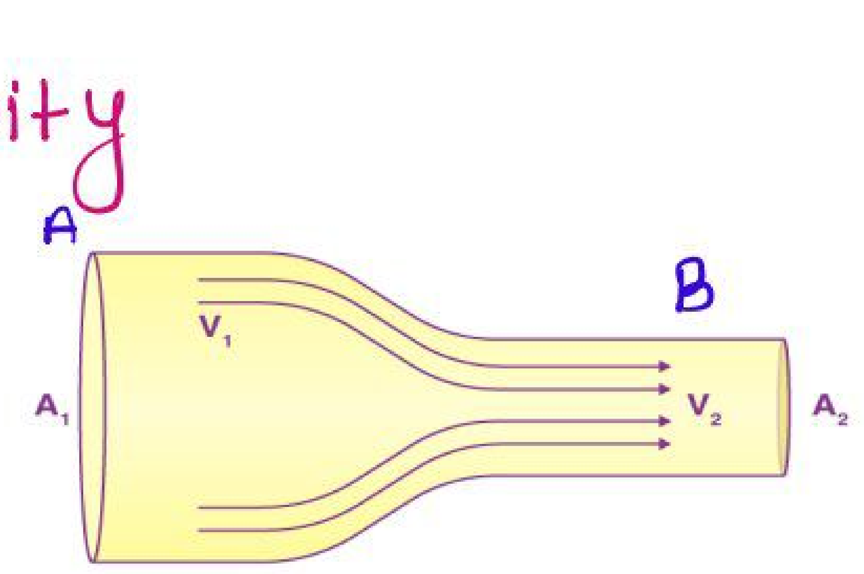


Fig. steady flow of liquid through a pipe of different cross sectional area

Consider the steady flow of non-viscous liquid through a pipe of varying cross-sectional area. Let A_1, V_1, ρ_1 be the cross sectional area, velocity of liquid and density of liquid at point A of the tube respectively. Similary A_2, v_2, ρ_2 be the cross sectional area, velocity of liquid and density of liquid at point B of the tube.

Volume of liquid entering per second at A = $A_1 \times v_1$

Mass of liquid entering per second at A = $A_1 \times v_1 \times \rho_1$

Similarly,

Mass of liquid leaving per second at B = $A_2 \times v_2 \times \rho_2$

If there is no loss of liquid in the tube and flow is steady, Then

$$A_1 \times v_1 \times \rho_1 = A_2 \times v_2 \times \rho_2$$

If the liquid is incompressible then density remain same $\rho_1 = \rho_2$

$$A_1 v_1 = A_2 v_2$$

$$\boxed{Av = \text{Constant}}$$

↳ This is called equation of continuity

This equation states that if the area of cross section of pipe becomes larger, the liquid's speed becomes smaller and vice versa. It is used in Bernoul's equation.

Energy of Liquid

1. Kinetic Energy (K.E)

$$K.E = \frac{1}{2} m v^2$$

$$KE \text{ per unit mass} = \frac{1/2\, m v^2}{m} = \frac{1}{2} v^2$$

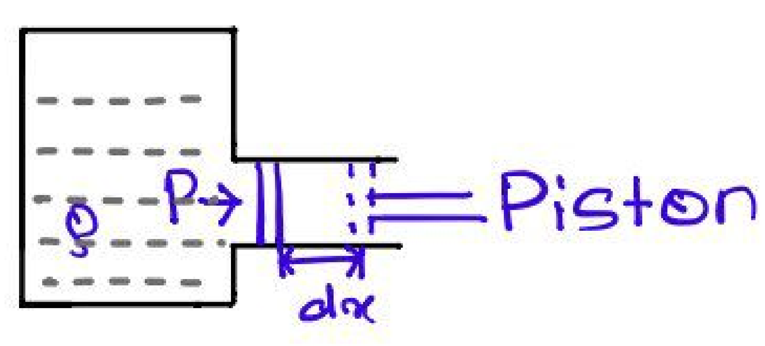
2. Potential Energy (P.E)

$$PE = mgh$$

$$PE \text{ per unit mass} = \frac{mgh}{m} = gh$$

3. Pressure energy

The energy possessed by liquid by virtue of its pressure is called pressure energy.



Force on piston = $P \times A$

If dx be small distance moved by the piston, then

work done = $PA \times dx = Pdv$

This work done is equal to the pressure energy of liquid,

Pressure energy for volume $dv = Pdv$

Mass of liquid having volume $dv = \rho dV$

Pressure energy per unit mass of liquid = $\frac{P\cancel{dv}}{\rho \cancel{dV}} = \frac{P}{\rho}$

$\therefore$ Total energy of liquid per unit mass = $\frac{1}{2}v^2 + gh + \frac{P}{\rho}$

$$\boxed{\therefore E = \frac{1}{2}v^2 + gh + \frac{P}{\rho}}$$

Bernoulli's Theorem

Statement:

For the streamline flow of an ideal fluid (incompressible and non-viscous) the total energy (pressure energy + Potential energy + kinetic energy) per unit mass remains constant at every cross-section throughout the flow.

$$\boxed{E = \frac{P}{\rho} + gh + \frac{v^2}{2} = \text{constant}}$$

Proof:

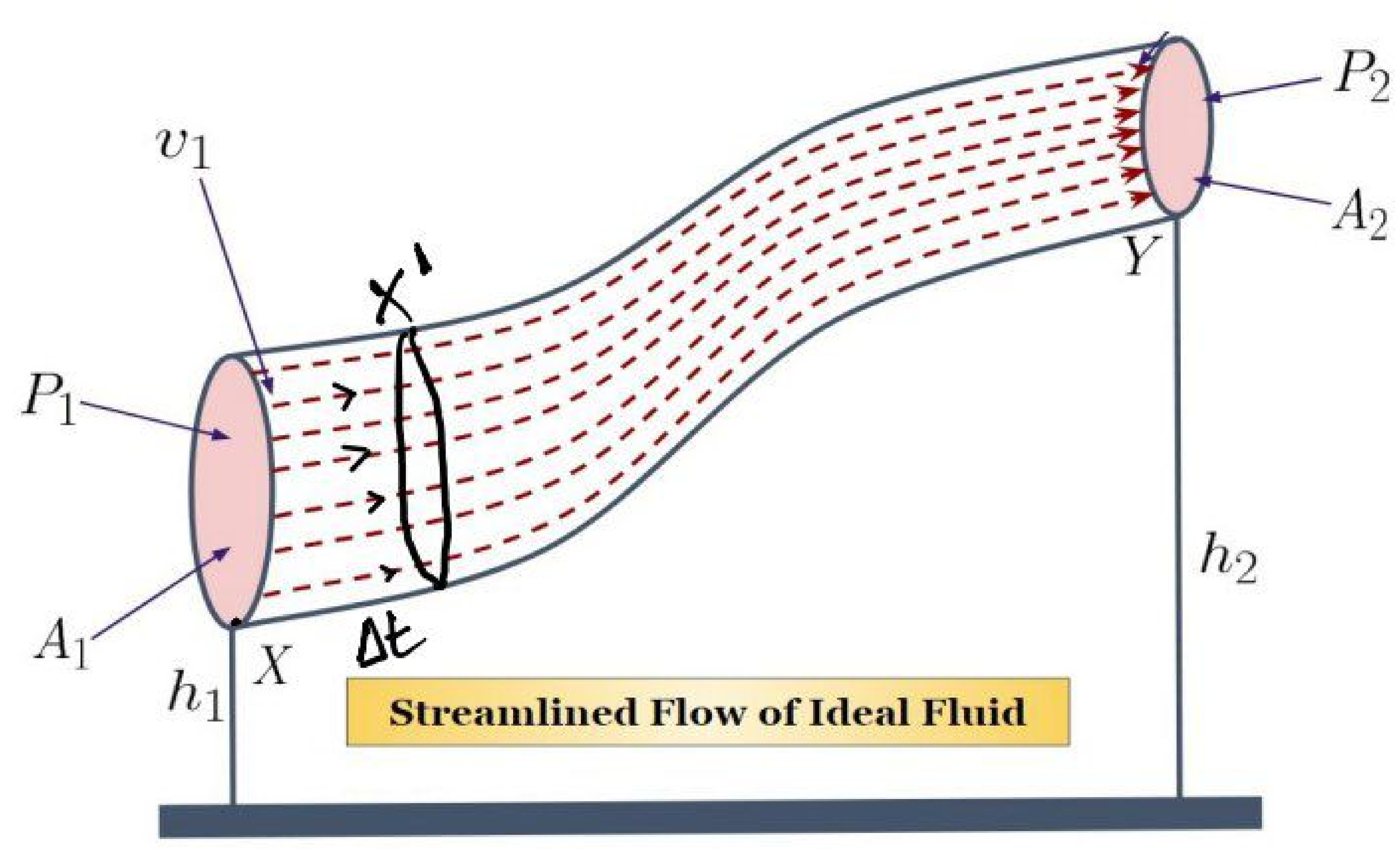


Fig. Bernoulli's Theorem

Consider a tube XY of varying area of cross-section through which an ideal liquid is in streamline flow. Let P_1, A_1, h_1, v_1 and P_2, A_2, h_2, v_2 be the pressure, area of cross section, height and velocity of flow at points X and Y respectively.

At point X,

Force acting = $P_1 A_1$

Distance travelled by liquid in time Δt = $v_1 \times \Delta t$

$\therefore$ Work done = Force × displacement

$= P_1 A_1 v_1 \Delta t$

when the liquid moves from X to X' in time Δt, the liquid at Y moves to Y' doing work against the pressure P_2 at Y at the same time.

Similarly,

Work done by the fluid against pressure P_2 at Y = $P_2 A_2 v_2 \Delta t$

The net work done on the liquid by the pressure energy in moving the liquid from X to Y,

$$W = P_1 A_1 v_1 \Delta t - P_2 A_2 v_2 \Delta t$$

From equation of continuity,

$$A_1 v_1 = A_2 v_2 = V = \frac{m}{\rho}$$

so,

$$W = P_1 V - P_2 V = (P_1 - P_2)\frac{m}{\rho}$$

where,

m = mass of liquid transferred

Increase in P.E of liquid (ΔPE) = $mgh_2 - mgh_1$ [$\because h_2 > h_1$]

Increase in K.E of liquid (ΔKE) = $\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$ [$\because a_1 > a_2$, $v_2 > v_1$]

According to Work Energy Theorem,

Work done = $\Delta PE + \Delta KE$

or, $(P_1 - P_2)\frac{m}{\rho} = mg(h_2 - h_1) + \frac{1}{2}m(v_2^2 - v_1^2)$

or, $\frac{P_1}{\rho} - \frac{P_2}{\rho} = gh_2 - gh_1 + \frac{1}{2}v_2^2 - \frac{1}{2}v_1^2$

or, $\frac{P_1}{\rho} + gh_1 + \frac{1}{2}v_1^2 = \frac{P_2}{\rho} + gh_2 + \frac{1}{2}v_2^2$

In general,

$$\frac{P}{\rho} + gh + \frac{v^2}{2} = \text{constant}$$

In terms of per unit volume

$$P + \rho gh + \frac{\rho v^2}{2} = \text{constant}$$

When a liquid flows through a horizontal pipe, the height $h_1 = h_2$ then equation become,

$$\frac{P_1}{\rho} + \cancel{gh_1} + \frac{1}{2}v_1^2 = \frac{P_2}{\rho} + \cancel{gh_1} + \frac{1}{2}v_2^2$$

or, $$\frac{P_1}{\rho} + \frac{1}{2}v_1^2 = \frac{P_2}{\rho} + \frac{1}{2}v_2^2$$

$$\boxed{\therefore \frac{P}{\rho} + \frac{v^2}{2} = \text{constant}}$$

Applications:

1. Atomiser or sprayer

When the rubber bulb B at the end of pipe is squeezed, the air blows in the tube T with high speed. According to Bernoulli's theorem, when the air blows in tube T, the pressure in it becomes less than the pressure in vessel in vessel R.

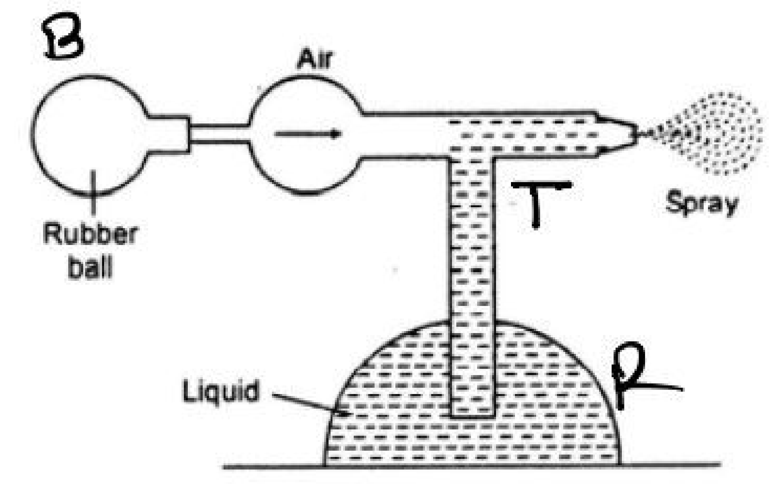


Fig. Atomiser and spray

Due to this, the liquid rises up in the tube T and is pushed out with air through the nozzle 'N' in the form of spray.

2. Flow meter / Venturi meter

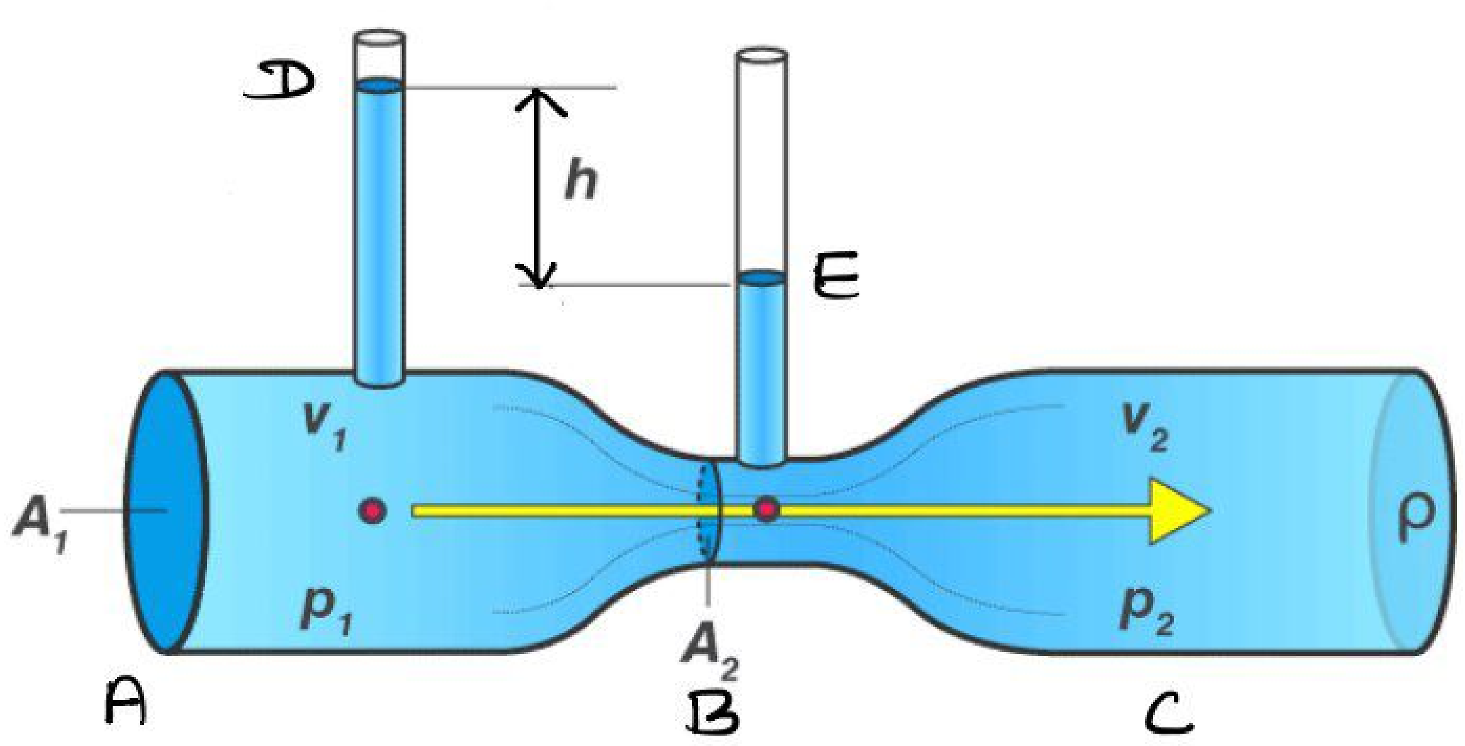


Fig. Flow meter or Venturimeter

It consists of two tubes A and B connected by narrow tube B. Using two tubes D and E, the difference in pressure of the liquid flowing through A and B can found out. Let A_1, P_1, v_1 and A_2, P_2, v_2 be the cross-sectional area, pressure and velocity of the liquid in the tube A and B respectively.

The volume of liquid flowing per second through the tube is given by,

$$V = A_1 v_1 = A_2 v_2$$

Since the pipe is horizontal, $h_1 = h_2$ and applying Bernoulli's theorem, we get

$$\frac{P_1}{\rho} + \frac{v_1^2}{2} = \frac{P_2}{\rho} + \frac{v_2^2}{2}$$

or, $$\frac{P_1}{\rho} - \frac{P_2}{\rho} = \frac{v_2^2}{2} - \frac{v_1^2}{2}$$

or, $$\frac{P_1 - P_2}{\rho} = \frac{v_2^2 - v_1^2}{2}$$

$$\left[\because P = \frac{F}{A} = \frac{mg}{A} = \frac{\rho \times V \times g}{A} = \frac{\rho \times \cancel{A} \times h \times g}{\cancel{A}} = \rho h g\right]$$

or, $$P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2)$$

or, $$\cancel{\rho} gh = \frac{1}{2}\cancel{\rho}\left[\frac{V^2}{A_2^2} - \frac{V^2}{A_1^2}\right]$$

or, $$gh = \frac{1}{2}\left(\frac{V^2 A_1^2 - V^2 A_2^2}{A_1^2 . A_2^2}\right)$$

or, $$gh = \frac{V^2(A_1^2 - A_2^2)}{2A_1^2 . A_2^2}$$

or, $$V^2 = \frac{2gh A_1^2 . A_2^2}{(A_1^2 - A_2^2)}$$

or, $$V = \sqrt{\frac{2gh A_1^2 . A_2^2}{(A_1^2 - A_2^2)}}$$

$$\boxed{\therefore V = A_1 . A_2 \sqrt{\frac{2gh}{A_1^2 - A_2^2}}}$$

Hence, the rate of flow of liquid can be calculated by measuring h, since A_1, A_2 are known for given venturimeter.

3. Lift on an aeroplane

The shape of the aeroplane wings is slightly convex upward and concave downward. Therefore, the speed of air above the wings become more than the below the wings. So, the pressure below the wings is more than that above the wings. Due to this difference in pressure, a vertical lift acts on an aeroplane.

4. Spinning ball

If a tennis ball is hit at an angle by bat, it spins as it travels in air and experiences a sideways force which causes it to curve in flight. This is due to air being dragged round by the spinning ball, increasing the airflow in one side and decreasing on other side. A pressure difference is thus created.